Journal of the Korean Nuclear Society
Volume 16, Number 1, March 1984

A Stream Line Method to Remove Cross Numerical Diffusion and Its Application to The Solution of Navier-Stokes Equations

Soon Heung Chang
Korea Advanced Institute of Science and Technology
(Received January 11, 1984)

교차수치확산을 제거하는 Stream Line방법과 Navier-Stokes방정식의 해를 위한 적용

장 순 흥
한국과학기술원
(1984. 1. 11 접수)

Abstract

The reduction of the truncation error including numerical diffusion, has been one of the most important tasks in the development of numerical methods. The stream line method is used to cancel cross numerical diffusion and some of the non-diffusion type truncation error. The two-step stream line method which is the combination of the stream line method and finite difference methods is developed in this work for the solution of the governing equations of incompressible buoyant turbulent flow. This method is compared with the finite difference method. The predictions of both classes of numerical methods are compared with experimental findings. Truncation error analysis also has been performed in order to the compare truncation error of the stream line method with that of finite difference methods.

요 약

수치확산을 포함한 truncation오차의 줄임은 수치해석의 중요한 과제가 되어왔다. Stream line방법이 교차수치 확산과 비확산형의 truncation 오차를 제거하기 위하여 고안되었다. 또한, stream line방법과 유한 차분법이 합쳐진 2단계 stream line방법이 비압축성 난류유동의 지배 방정식을 풀기위하여 고안되었다. 이 방법은 유한 차분법과 비교되었으며, 두 방법 모두 실험자료와 비교되었다. 그리고, 두 방법의 truncation 오차를 비교하기 위하여 truncation 오차 분석이 행해졌다

1. Introduction

Computer codes used in the thermalhydraulic analysis involve the discretization of a set of differential equations. There is a truncation error which is the difference between the algebraic set of equations constituting the numerical approximations and that of the original differential equations. This truncation error degrades the accuracy of a calculation. As a result of this error, thermal-hydraulic computer predictions generally describe more diffusion and mixing than true solutions. The reduction of the truncation error, including numerical diffusion, has been one of the most important tasks in the development of numerical methods. To show the truncation error and numerical diffusion in two dimension, a Taylor series analysis was performed on the convection equation of x-direction momentum. The detailed derivation of the analysis is presented in Appendix. If we retain only the terms containing u_{xx}, u_{yy}, and u_{xy} from the result in Appendix, we obtain a diffusive error of the form.

$$\begin{array}{c}\text{False} \\ \text{diffusion}\end{array}\left(\begin{array}{c}\text{Numerical} \\ \text{diffusion}\end{array}\right)=\left[\frac{-\Delta t}{2}u^2+\frac{\Delta x}{2}|u|\right]u_{xx}+\left[\frac{-\Delta t}{2}v^2+\frac{\Delta y}{2}|v|\right]u_{yy}-uv\Delta t u_{xy} \quad (1)$$

There are also various non-diffusive truncation error terms such as $\Delta t\cdot u\cdot u_y\cdot v_x$ and $\Delta t\cdot u\cdot u_x^2$. It must be noticed in a single dimension that the trnncation error of the first order is always of the diffusion type. Therefore in a single dimension, there is no lower non-diffusion-type truncation error.

It is also important to notice from Eq.1 that there is a cross flow numerical diffusion term, $-uv\Delta t\ u_{xy}$, arising in the two dimensions.This cross numerical diffusion occurs when flow is oblique to the grid lines and when there is a non-zero gradient of the dependent variable [1]. Eq. 1 also gives the magnitude of the cross numerical diffusion coefficient as

$$\begin{aligned}\sigma_{cn}&=|uv|\Delta t, \text{ or}\\ &=U^2\cos\theta\sin\theta\Delta t, \text{ or}\\ &=\frac{U^2}{2}\sin 2\theta\Delta t \qquad (2)\end{aligned}$$

where u is the resultant velocity ($U^2=u^2+v^2$) and θ is the angle (between 0° and 90°) made by the velocity vector with the x-direction.One can see from this equation that no cross numerical diffusion is present when the resultant flow is parallel to one of the sets of grid lines.

2. The stream line method

The stream line method solves the following convection equation.

$$Q_t+uQ_x+vQ_y=0 \qquad (3)$$

where the initial condition is $Q(x,y,0)=Q_0(x,y)$. The physical meaning of Eq. 3 is that the convected qualtity Q remains constant along the characteristic line defined by $\frac{dx}{dt}=u$ and $\frac{dy}{dt}=v$. The basis of the stream line method of solving the convection equation consists of the determination and trace of the stream line. Let the transported quantity, Q, be known at each node at time t_0. Then, the goal of the solution is to calculate Q at each node at the next time step, $t_0+\Delta t$. From Figure 2, consider a particle which is at a node $A(A_x, A_y)$ at time $t_0+\Delta t$. To calculate Q $(A, t_0+\Delta t)$, the position $P(P_x, P_y)$ where the particle was at time t_0 must be found using the equations.

$$P_x=A_x-\int_{t_0}^{t_0+\Delta t} u\{\xi(t')\}\,dt' \qquad (4)$$

$$P_y=A_y-\int_{t_0}^{t_0+\Delta t} v\{\xi(t')\}\,dt' \qquad (5)$$

where ξ is the stream line. After locating the position $P, Q(A, t_0+\Delta t)$ can be calculated using the relation $Q(A, t_0+\Delta t)=Q(P, t_0)$. There is a problem, however, in that P is not generally a discretization node. Thus, the value $Q(P, t_0)$ must be interpolated. Therefore, the solution procedure of the stream line method can be summarized as consisting of two steps. One is evaluation of the characteristic integrals of Eqs. 4 and 5. The other is numerical interpolation for evaluation of dependent quantity, Q.

In one dimension, the characteristic integral evaluation proceeds as follows: In Fig. 1, displacement $\vec{A}-\vec{P}$ is equal to the characteristic integral. If the direction of velocity is upward, the characteristic integral, CI, becomes:

$$CI=\vec{A}-\vec{P}=u_p\Delta t \qquad (6)$$

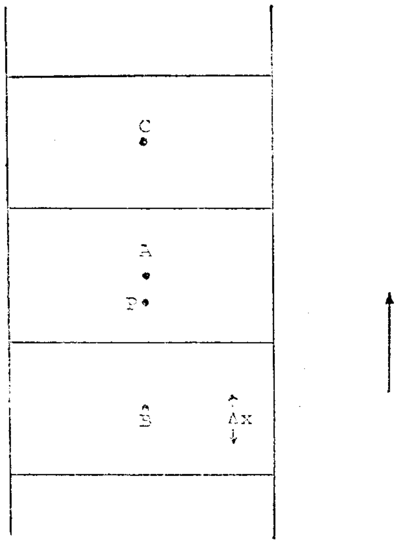

Fig. 1. One Dimensional Mesh Configuration A, B, C and D indicate the nodes (i, j), (i-l, j), (i-l, j-l) and (i, j-l), respectively.

Since P is located between $\vec{A}$ and $\vec{B}$ subject to the Courant condition $\left(\frac{u\delta t}{\delta x}<1\right)$ the following relationship holds:

$$u_p=u_A+\frac{u_B-u_A}{\Delta x}(\vec{P}-\vec{A}) \tag{7}$$

Combining Eq. 6 with Eq. 7 yields the result:

$$CI=\left|\frac{\Delta t u_A}{\Delta t(u_A-u_B)-\Delta x}\right|\Delta x \tag{8}$$

From Eq. 8 it is demonstrated that $CI=-u_A\Delta t=-u_B\Delta t$ if $u_B=u_A=$ constant.

In two dimension, the characteristic integral evaluation proceeds as follows: Applying Eqs. 4 and 5 in Fig. 2 yields:

$$\text{CIX}=A_x-P_x=u_p\Delta t, \tag{9}$$

$$\text{CIY}=A_y-P_y=v_p\Delta t \tag{10}$$

Since it is impossible to find exactly u_p and v_p, it is necessary to approximate u_p and v_p. The simplest way to approximate them is to make $u_p\cong u_A$, and $v_p\cong v_A$. An improvement upon the simplest approximation is to approximate u_p and v_p by the axis approximation, to make

$u_p\cong u_p'$ and $v_p\cong v_p'$ or to make

$u_p\cong u_p''$ and $v_p\cong v_p''$ as shown in Fig. 2.

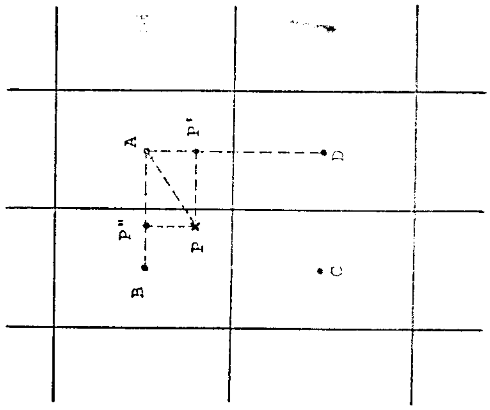

Fig. 2. Two Dimensional Mesh Configuration

If we approximate $u_p=u_p'$ and $v_p=u_p'$, Eqs. 9 and Eq. 10 yield:

$CIX=u_p'\Delta t$, and

$CIY=v_p'\Delta t$

The value of CIY can be calculated in the same way as in the same way as in one dimension. Using Eq. 8 between A and P yields:

$$CIY=\left[\frac{\Delta t v_A}{\Delta y-\Delta t(v_A-v_D)}\right]\Delta y \tag{11}$$

By using the calculated value of CIY, CIX can be calculated as follows:

$$CIX=u_p'\Delta t=\left\{u_A+\frac{CIY}{\Delta y}(u_D-u_A)\right\}\Delta t \tag{12}$$

Eventually, CIX and CIY can be calculated, using Eqs. 11 and 12, respectively. The same principle applies to the calculation of CIX and CIY when approximating $u_p\cong u_p''$ and $v_p\cong v_p''$. The only difference lies in calculating CIX first and CIY next.

As mentioned previously the position P in Figs. 1 and 2 is not a discretization node in most cases. Thus, an interpolation is necessary in order to calculate the value of $Q(P,\ t_0)$.

In this study, linear interpolation and the second order interpolation have been considered. Only linear interpolation has been adopted for implementation in single dimensional and two dimensional calculations. This is because this interpolation is physically reasonable, simple to apply and permits one to reduce the running time. The linear interpolation in single dimension is the following. Applying the interpolation to Fig. 1 yields the following relation:

$$Q_p = Q_A - \frac{CIX}{\Delta x}(Q_B - Q_A) \quad (13)$$

The linear interpolation in two dimensions can be represented as shown in Fig. 2 yielding the result

$$Q_p = (1-m)(1-n)Q_A + m(1-n)Q_B + (1-m)nQ_D + mnQ_C,$$

where $m = -CIX/\Delta x$, $n = -CIZ/\Delta Z$. (14)

3. Truncation Error Analysis of The Stream Line Method

Assume flow is upward in one dimension as shown in Fig. 1. Since v is constant from the continuity equation, the characteristic integral, CI, becomes $v\Delta t$. Linear interpolation, subject to the Courant condition, yields:

$$Q_i^{n+1} = Q_i^n + \left(\frac{Q^n_{i-1} - Q_i^n}{\Delta x}\right) v\Delta t \quad (15)$$

From Eq. 15, it is seen that there exists an equivalence between the donor cell method and the stream line method with linear interpolation in one dimension.

In order to make a comparison to the two dimensional stream line method, the two dimensional donor cell method is examined. In two dimensions, the donor cell method is stated as:

$$Q_{i,j}^{n+1} = Q^n_{i,j} + \frac{u\Delta t}{\Delta x}(Q^n_{i-1,j} - Q^n_{i,j}) + \frac{v\Delta t}{\Delta z}(Q^n_{i,j-1} - Q^n_{i,j}) \quad (16)$$

where $u = u^n{}_A > 0$. $v = v^n{}_A > 0$. It is assumed the flow is upward and from left to right. It is also assumed that $u_p = u$ and $v_p = v$, which is the simplest approximation for the characteristic integral in Fig. 2. Then the characteristic integrals, CIX and CIZ, become:

$$CIX = -u_A\Delta t, \text{ and}$$
$$CIZ = -v_A\Delta t$$

Inserting CIX and CIZ into Eq. 14 yields the result

$$Q_{i,j}^{n+1} = Q^n_{i,j} + \frac{u\Delta t}{\Delta x}(Q^n_{i-1,j} - Q^n_{i,j}) + \frac{v\Delta t}{\Delta z}(Q^n_{i,j-1} - Q^n_{i,j}) + \frac{uv\Delta t^2}{\Delta x\Delta z}(Q^n_{i,j} - Q^n_{i-1,j} - Q^n_{i,j-1} + Q^n_{i-1,j-1}) \quad (17)$$

The last term of Eq. 17 is equivalent to the finite difference form of the term $uv\Delta t^2 Q_{xz}$. As is pointed out previously, $uv\Delta t^2 Q_{xz}$ is from the cross numerical diffusion term. From this fact, it can be concluded that the cross numerical diffusion can be cancelled by using the stream line method in two dimensions.

By using the axis velocity approximation, $u_p = u_p'$ and $v_p = v_p'$ (or $u_p = u_p''$ and $v_p = v_p''$) in Fig. 2, CIX and CIZ are seen to be

$$CIX = \Delta t(u - u_x \cdot u\Delta t), \text{ and}$$
$$CIZ = \Delta t(v - v_x \cdot u\Delta t) \quad (18)$$

Inserting Eq. 18 into Eq. 14 yields Q_{ij}^{n+1} as

$$Q_{i,j}^{n+1} = Q^n_{i,j} + \frac{u\Delta t}{\Delta x}(Q^n_{i-1,j} - Q^n_{i,j}) + \frac{v\Delta t}{\Delta z}(Q^n_{i,j-1} - Q^n_{i,j}) + \frac{uv\Delta t^2}{\Delta x\Delta z}(Q^n_{i,j} - Q^n_{i-1,j} - Q^n_{i,j-1} + Q^n_{i-1,j-1}) + u\Delta t^2 u_x \frac{Q^n_{i-1,j} - Q^n_{i,j}}{x} + u\Delta t^2 v_x \frac{Q^n_{i,j-1} - Q^n_{i,j}}{\Delta z} \quad (19)$$

The last three terms of Eq. 19 are equivalent to $uv\Delta t^2 Q_{xz}$, $\Delta t^2 u\ u_x Q_x$ and $u\Delta t^2 v_x Q_x$. From Eq. A. 8 in Appendix, it is shown that these three terms are truncation errors from the time derivative term. The term $uv\Delta t^2 Q_{xz}$ is a diffusive type truncation error. The other two terms are, however, non-diffusion type truncation errors.

From this fact, it can be concluded that non diffusive type truncation errors other than cross numerical diffusion errors can be reduced by using the stream line method with linear interpolation and axis approxmation for the characteristic integral.

4. Application of The Stream Line Method to Navier-Stokes Equations

The classical characteristic method has been used for the solution of quasi-linear hyperbolic differential equations, governing compressible flow and the flow of small compressibility where the sonic characteristic lines are considered. The isentropic relation$\left(\frac{\partial P}{\partial \rho_s}\right)=C^2$ together with the continuity equation gives the pressure equation. Then, the pressure can be calculated along the sonic characteristics line.

There are the major difficulties in the application of classical characteristics methods to turbulent incompressible flow which concerns this study. First,sonic characteristics cannot be used to calculate pressure because the sonic velocity in incompressible flow is infinite. Secondly, the method of characteristic is basically designed to solve the hyperbolic convection equation.

This method cannot deal with the effect of turbulent diffusion and some effects other than convection. To overcome these two difficulties, the two step stream line method is designed. To solve the first difficulty,the two step stream line method solves for the pressure implicitly. To resolve the second difficulty, the transport equation is split into two equations where the stream line method is only applied to the convection equation.

Fig. 3 describes how to apply the stream line method to the solution of Navier-Stokes equation.

The application can be explained through the following Navier-Stokes equation, Eq. 20 and Continuity equation, Eq. 21

$$Q_t+C=D+B+P, \tag{20}$$

$$\nabla \cdot Q=0, \tag{21}$$

where Q=velocity variable

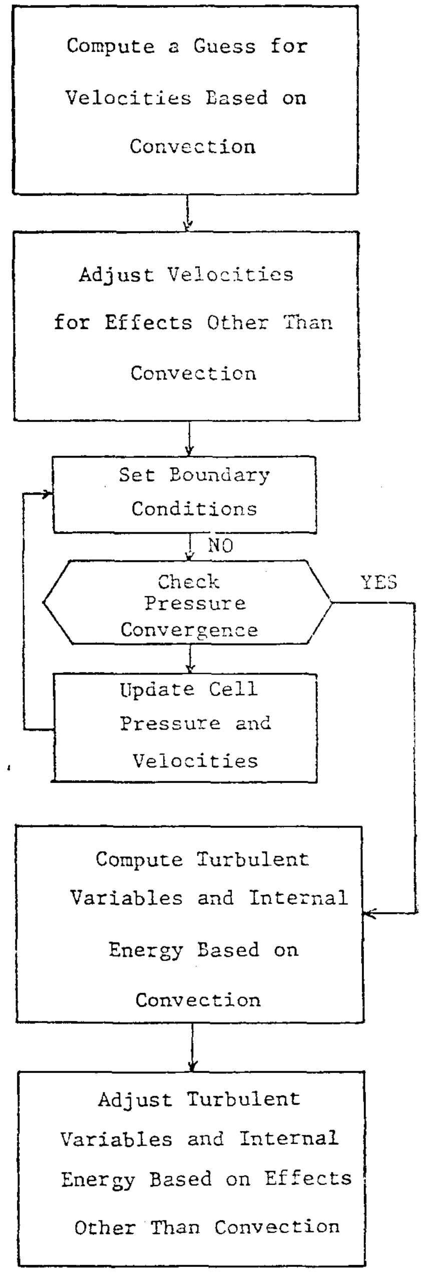

Fig. 3. Flow Diagram of Solution Scheme of the Two Step Stream Line Method

Q_t=time derivative term
C=convection term
D=diffusion term
B=buoyancy term
P=pressure gradient term.

The solved velocities must satisfy both Eqs. 20

and 21. To implement the two step stream line method, Eq. 20 is split into two equations such as Eqs. 22 and 23.

$$\frac{\bar{Q}-Q^{n}}{\Delta t}+C=0 \tag{22}$$

$$\frac{\tilde{Q}^{n+1}-\bar{Q}}{\Delta t}=D+B+P \tag{22}$$

Eq. 22 is the convection equation of momentum which is solved through the stream line method. This gives the intermediate value of the approximations of new velocity. Eq. 23 is solved through the finite difference method. But for the calculation of the density for use in the buoyancy term, the convected internal energy using the velocities at previous time step is used to better stabilize the calculation. The solution of Eq. 23 is the approximation for new velocity. This explicity calculated new velocity does not satisfy Eq. 21 the continuity equation. Therefore the new velocity must be adjusted to satisfy Eq. 21 by making appropriate change in the cell pressure using the SMAC method [2]. When convergence has been achieved, compute the intermediate values of the internal energy and turbulence transport variables for a mesh by considering only convection with the velocities of each new time step through the stream line method. With these intermediate values, the internal energy and turbulence variables of each new time step is computed by considering the effects other than convection through finite difference methods.

In order to validate and test the developed numerical method, a comparison between the predictions of the computer code and the experimental results on the field is performed. The experimental data of Howard and Carbajo [5] on thermal hydraulic behavior in Liquid Metal Fast Breeder Reactor (LMFBR) outlet plenum are used for this study. Among twelve tests, two tests, BT 16 and BT 21, are chosen for the comparison. The plennm fluid in test BT 16. before the flow of cold water was initiated,

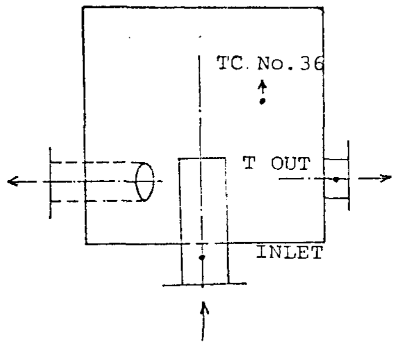

Fig. 4. Outlet Plenum Configuration of Test BT 16

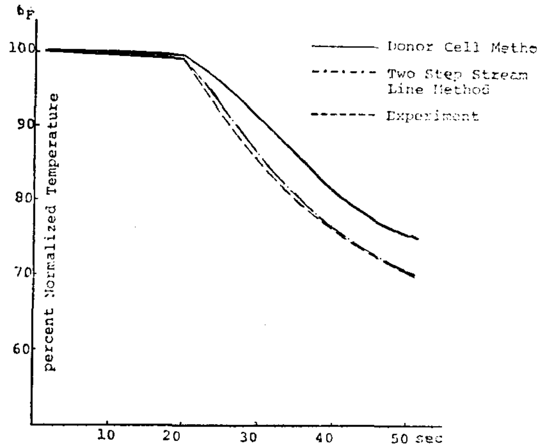

Fig. 5. Experimental and Predicted Temperature of Outlet Nozzle in Test BT 16

was hot and stgnant. In test BT 21, the flow was recirculating in a steady state condition before the flow coastdown transient became effective.

Fig. 5 shows the transient temperature prediction at the outlet nozzle for test BT 16. Fig. 6 shows predictions and experimental results for the temperature of the thermocouple No. 36 in test BT 16. (Refer to Fig. 4 for the outlet plenum configuration of test BT16) Fig. 7 shows the predictions and experimental result for the temperature of the reactor outlet nozzle in test BT 21.

As seen in Figs. 5, 6 and 7 the predictions by the developed two step stream-line method are closer to the experimental results than those

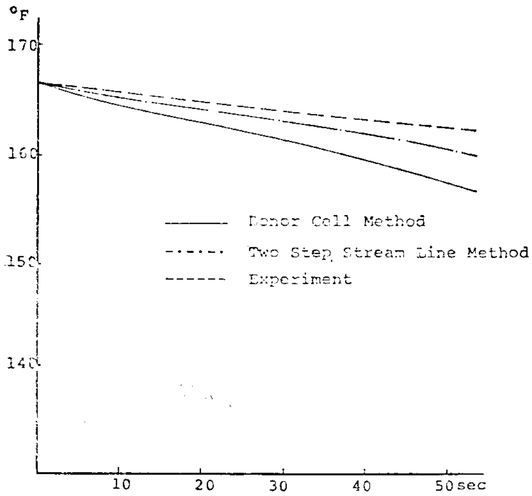

Fig. 6. Experimental and Predicated Temperature of Thermocouple No. 39 in Test BT 16

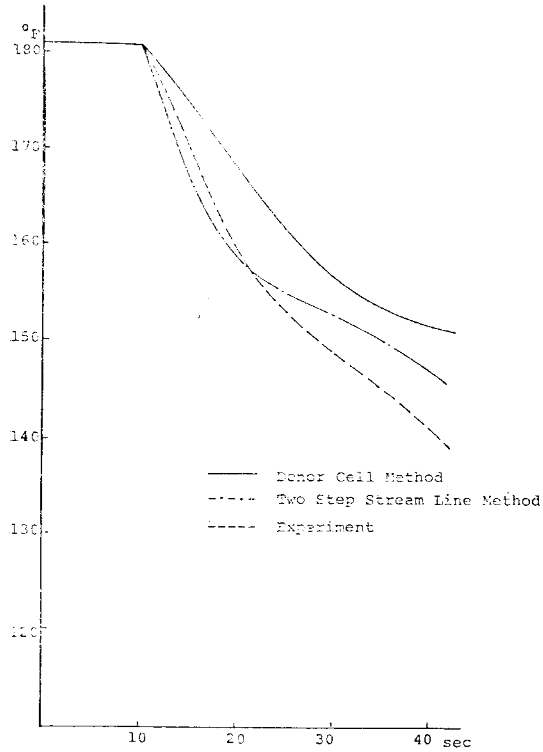

Fig. 7. Experimental and Predicted Temperature of Outlet Nozzle in Test BT 21

by the finite difference donor cell method.

5. Summary and Conclusions

1) The two step stream line method which is the combination of the stream line method and the finite difference method is developed as a numerical method to reduce truncation error including numerical diffusion for the solution of Navier-Stokes equations, energy equation, the turbulence transport equations and continuity equation which govern the behavior of two dimensional incompressible nonisothermal turbulent flow.

2) The developed two step stream line method gives better agreement with experimental findings than the finite difference donor cell method.

3) Trunction error analysis shows that the stream line method with linear interpolation is equivalent to the finite difference donor cell method in one dimension.

Truncation error analysis also shows that the stream line method cancels cross flow numerical diffusion and some of the non-diffusion type truncation errors which the finite difference donor cell method has in two dimension.

References

1. Patankar, S.V., "Numerical Heat Transfer and Fluid Flow", Mc-Graw-Hill, New York, 1980.
2. Amsden, A.A., Harlow, F.H., "The SMAC Met hod: A Numerical Technique for Calculating Inc ompressible Flows," Los Alamos Sci. Lab.Report LA-4370, May 1970.
3. Chang, S.H., "Comparative Analysis of Numerical Methods For the Solution of Navier-Stokes Equation," MIT Ph. D Thesis, 1981.
4. Tang, Y.W., et al., "Thermal Analysis of Liquid Metal Fast Breeder Reactors," American Nuclear Society pp. 164-170, 1978.
5. Howard, P.A., Carbajo, J.J., "Experimental Study of Scram Transients in Generalized Liquid-Metal Fast Breeder Reactor Outlet Plenums," Nuclear Technology, Vol. 44, July 1979.
6. Benque, J.P., et al., "A Finite Element Method for Navier-Stokes Equation," Laboratorie National d'Hydraulique, Electricite de France.

Acknowledgement

The financial support of the U.S. Department of Energy is gratefully acknowledged.

Appendix
Taylor Series Expansion of X-Direction Momentum Convection Equation in two Dimensions

The expansion has been made under the following assumptions:

- Single phae flow
- Incompressibe
- Uniform mesh spacings and time intervals
- All flow is Upward and to the Right, i.e.,

$u_{i+1/2}\geq 0 \quad v_{i+1/2}\geq 0$

Under these conditions the momentum in the x-direction may be written as:

$$\frac{(u^{n+1}-u^n)_{i+1/2}}{\Delta t}+\Big[u_{i+1/2}\Big(\frac{\Delta u}{\Delta x}\Big)^n_{i+1/2}+v^n_{i+1/2}\Big(\frac{\Delta u}{\Delta y}\Big)^n_{i+1/2}\Big]=0$$

Expanding all finite difference quantitites in Taylor Series about the points i and n up to terms in Δx^2 and Δt^2 yields:

$$u^{n+1}_{i+1/2}=u+\frac{\Delta x}{2}u_x+\Delta t u_t+\frac{\Delta x^2}{8}u_{xx}+\frac{\Delta x\Delta t}{2}u_{xt}+\frac{\Delta t^2}{2}u_{tt}+0(\Delta x^3, \Delta t^3),$$

$$u_{i+1/2}{}^n=u+\frac{\Delta x}{2}u_x+\frac{\Delta x^2}{8}u_{xx}+0(\Delta x^3).$$

Thus, the following relation can be obtained:

$$\frac{(u^{n+1}-u^n)_{i+1/2}}{\Delta t}=u_t+\frac{\Delta x}{2}u_{xt}+\frac{\Delta t}{2}u_{tt}+0(\Delta t^2). \tag{A.1}$$

For $u^n_{i+1/2}\geq 0$, the following relations hold

$$\Big(\frac{\Delta u}{\Delta x}\Big)^n_{i+1/2}=(u^n_{i+1/2}-u^n_{i-1/2})/\Delta x,$$

$$u^n_{i+1/2}=u+\frac{\Delta x}{2}u_x+\frac{\Delta x^2}{8}u_{xx}+0(\Delta x^3),$$

$$u^n_{i-1/2}=u-\frac{\Delta x}{2}u_x+\frac{\Delta x^2}{8}u_{xx}+0(\Delta x^3).$$

Thus, the following relation can be obtained:

$$u^n_{i+1/2}\Big(\frac{\Delta u}{\Delta x}\Big)^n_{i+1/2}=uu_x+\frac{\Delta x}{2}u^2{}_x+\frac{\Delta x^2}{8}u_xu_{xx}+0(\Delta x^3). \tag{A.2}$$

For $v_{i+1}\geq 0$, the following relation can be obtained in the same way:

$$v_{i+1/2}\Big(\frac{\Delta u}{\Delta y}\Big)^n_{i+1/2}=vu_y+\frac{\Delta x}{2}v_xu_y+\frac{\Delta y^2}{8}u_yv_{yy}+\frac{\Delta x^2}{2}u_yv_{xx}-\Big[\frac{\Delta y}{2}v+\frac{\Delta y\Delta x}{4}v_x\Big]u_{yy}+\Big[\frac{\Delta x}{2}v+\frac{\Delta x^2}{4}v_x\Big]u_{xy}+0(\Delta x^3, \Delta_y{}^3). \tag{A.3}$$

Substitution of Eqs. A.1, A.2 and A.3 into the finite difference x-momentum convection equation yields:

$$u_t+uu_x+vu_y+\Big[\frac{\Delta t}{2}u_{tt}+\frac{\Delta x}{2}u_{xt}+\frac{\Delta x}{2}u_x{}^2+\frac{\Delta x^2}{8}u_xu_{xx}+\frac{\Delta x}{2}v_xu_y+\frac{\Delta y^2}{8}u_yv_{yy}+\frac{\Delta x^2}{2}u_yv_{xx}-\frac{\Delta y}{2}vu_{yy}-\frac{\Delta x}{4}\Delta yv_xu_{yy}+\frac{\Delta x}{2}vu_{xy}+\frac{\Delta x^2}{4}v_xu_{xy}\Big]=0$$

Denoting the terms in braces by R, the above equa tion bebomes:

$$u_t+uu_x+vu_y+R=0.$$

If $R=0$ the above equation becomes the original x-momentum equation, so R is a residual error term introduced by the differencing scheme. Two momentum equation and continuity equation and contiuity equation are:

$$u_t=-uu_x-vu_y,$$

$$v_t=-uv_x-vv_y,$$

$$u_x+v_y=0 \tag{A.4}$$

The above equations yield the following relations:

$$u_{tt}=-u_tu_x-uu_{xt}-v_tu_y-vu_{yt}, \tag{A.5}$$

$$u_{xt}=(u_t)_x=-u_x{}^2-uu_{xx}-v_xu_y-vu_{xy}, \tag{A.6}$$

$$u_{yt}=(u_t)_y=-u_yu_x-uu_{xy}-v_yu_y-vu_{yy}. \tag{A.7}$$

Substituting equations A.4, A.5, A.6 and A.7 into R and rearranging yields:

$$R=\Big[\frac{\Delta t}{2}u^2-\frac{\Delta x}{2}u\Big]u_{xx}+\Big[\frac{\Delta t}{2}v^2-\frac{\Delta y}{2}v\Big]u_{yy}+uv\Delta tu_{xy}+\frac{\Delta x^2}{2}u_yv_{xx}+\frac{\Delta y^2}{8}u_yv_{yy}+\Delta tu\Big[u_x{}^2+u_yv_x\Big]+\frac{\Delta x^2}{2}v_xu_{xx}-\frac{\Delta x}{4}\Delta yv_xu_{yy}.$$

This then is the truncation error, to orddr Δx^2, Δy^2 and Δt^2, introduced by the explicit upward differencing scheme for the x-momentum convection equation in two dimensions.