Journal of the Korean Nuclear Society
Volume 8, Number 4, December, 1976

Finite Element Computation of Slab Criticality and Milne Problem

Chang Hyo Kim and Jong Hwa Chang

College of Engineering, Seoul National University

Dong Hoon Kim

Reactor Physics Division, Korea Atomic Energy Research Institute

(Received September 10, 1976)

Abstract

A finite element method is formulated for one-speed integral equation for the neutron transport in a slab reactor. The formulation incorporates both the linear and the cubic Hermite interpolating polynomials and is used to compute the approximate solutions for the slab criticality and Milne problem. The results are compared with the exact solutions available and then the effectiveness of the method is extensively discussed.

요 약

슬레브형로내의 중성자수송에 관한 적분 방정식을 유한요소법으로 기술했다. Hermite 내삽에 의한 1차 및 3차 다항식을 사용하여 얻은 수식으로 슬레브형로의 임계치와 Milne 문제의 근사해를 계산하고 해석해와의 비교를 통하여 유한 요소법의 유용성을 논의했다.

1. Introduction

The finite element method has become a popular computational technique for the solutions of various neutronic problems such as multi-group diffusion equation[1-4] and neutron transport equation.[5-7] The method is closely tied up with the Ritz-Galerkin scheme utilizing as expansion functions the piecewise polynomials which are defined in the prescribed subregions of the domain of the independent variables.

In this paper we investigate the use of both the linear and the cubic Hermite interpolating polynomials[1] for obtaining the approximate solutions of an infinte slab criticality[8] as well as the Milne problem[9]. These problems are of interest because their exact solutions are available and thereby the quantitative comparision between the exact and the approximate solutions are possible.

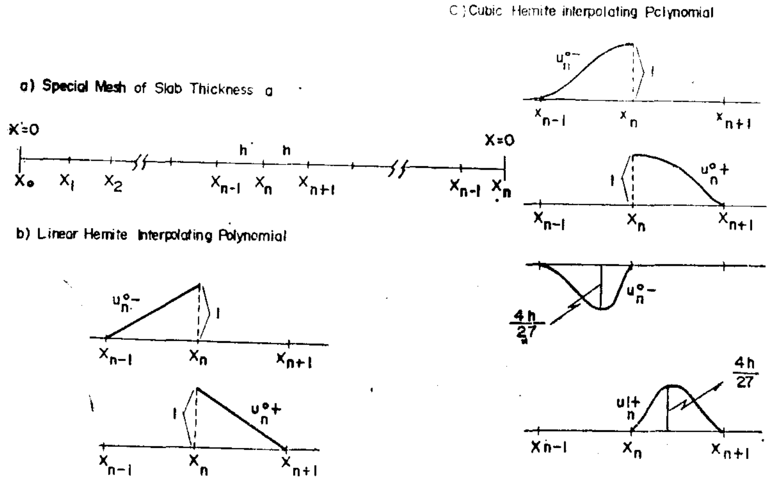

Fig. 1. Spatial Mesh of Slab Thickness a

2. Formulation

2.1 Slab Criticality

The one-speed integral equation for the neutron transport in a slab reactor is given by

$$\phi(x) = \frac{c}{2}\int_0^a E_1(1x-x'1)\ \phi(x')\,dx'. \qquad (1)$$

c is the average number of secondary neutrons per collision due to scattering and fission. $E_1(x)$ is the expontial integral function. The criticality problem to be considered here is to find the smallest c for a given slab of thickness a. The specific numerical method for the problem is a finite element method combined with the piecewise continuous Hermite interpolating polynomials.

As shown in Fig. 1, the slab is first-divided into N intervals. The Hermite interpolations are then defined in each interval, $(n\text{-}1)h \leq x \leq nh$. Let's assume that Eq. (1) is approximately satisfied by a trial function, $\phi_{tr}(x)$, represented by an expansion

$$\phi_{tr}(x) = \sum_{n\ 1}^{N}\sum_{p\ 0} \phi_n^p \cdot u_n^p(x). \qquad (2)$$

The runing index p is related to the order of the interpolating polynomials. For the linear interpolation, the summation over p is unnecessary. For the cubic interpolation, the summation over p goes from 0 to 1. The expansion coefficient ϕ_n^0, denotes neutron flux, while ϕ_n^1 neutron current, at the mesh point n. Combining Eq. (2) with Eq. (1) and apppiying the Ritz Galerkin scheme, one obtains a system of equations of the matrix form,

$$U\phi = \frac{c}{2}E\phi, \qquad (3)$$

U is a tri-diagonal matrix whose elements are given by

$$U_{ki}^{pp'} = (U_k^p, U_i^{p'}) = \int_0^a U_k^p(x)\,U_i^{p'}(x)\,dx. \qquad (4)$$

E is symmetric square matrix. Its elements are defined by

$$E_{ki}^{pp'} = \int_0^a dx U_k^p(x)\int_o^a E_1(|x-x|)\ U_i^{p'}(x')\,dx'. \qquad (5)$$

ϕ represents a column vector,

$$\phi = \text{column vector } \{\phi_0^o, \cdots\phi_0^p, \phi_1^o, \cdots\phi_1^p, \cdots\phi_N^p\} \qquad (6)$$

The explicit expressions for U, and E depend on the interpolationg polynomials adopted. They are evaluated in Appendix in terms of both the linear and cubic Hermite interpolating polynomials.

2.2 Milne Problem

This is the problem of finding the distribution of neutrons in an infinite half-space ($x \geq 0$) through which neutrons are diffusing from a source at infinity. The problem is often termed as that of a half-space with a source at infinity. The objective that we seek here is to test the accuracy of the finite element method with the use of Hermite interpolating polynomials. The same problem has been considered by Abu-Shumays et. al[10]. But they used the liniar and cubic spline approximations. For the case of linear approximation, our approach is the same as theirs, while two are different for the cubic approximation.

As the trial solution of the Milne problem, let us suppose that Eq. (1) is approximately satisfied by

$$\phi(x) = x_0 + x + \phi_{tr}(x). \tag{7}$$

x_0 is the extrapolation distance, 0.7104 in unit of the neutron mean free path. This form is used in Ref. 10 and is chosen to represent reasonably well the neutron distribution near the vacuum boundary. Subsituting Eq. (7) into Eq. (1), one finds the working formula to which the finite element method is applicable,

$$\phi_{tr}(x) = \frac{1}{2}\int_0 E_1(|x-x'|)\phi_{tr}(x')\,dx' - \frac{x_0}{2}E_2(x) + \frac{1}{2}E_3(x). \tag{8}$$

Assuming the expansion, Eq. (2), for $\phi_{tr}(x)$, one obtains

$$U\phi = E\phi + b. \tag{9}$$

U and E are defined in the same way as Eq's. (4) and (5), except that the upper limit of integral is replaced by the infinity. b is a column vector given by

$$b = \text{column vector } \{b_0^{\,o}, \cdots b_0^{\,p}, b_1^{\,o} \cdots b_1^{\,p} \cdots b_N^{\,p}\} \tag{10}$$

where

$$b_i^{\,p} = -\frac{x_0}{2}(E_2(x), U_i^{\,p}(x)) + \frac{1}{2}(E_3(x), U_i^{\,p}(x)). \tag{11}$$

The numerical accuracy can then be checked by the following quantities

$$l = \frac{\int_0^\infty \phi_{tr}(x)\,dx}{\phi_{tr}(0)}, \tag{12}$$

and

$$x = \frac{\int_0^\infty x\phi_{tr}(x)\,dx}{\int_0^\infty \phi_{tr}(x)\,dx}, \tag{13}$$

3. Numerical Results and Discussion

The slab criticality and the neutron flux near the vacuum boundary are very much dependent upon the evaluation of matrices, U and E. The computation of matrix U is straightforward and can be made with any accuracy. But that of matrix E is a little complicated and involves the truncation error. In appendix the cubic Hermite in terpolating polynomials and their explicit expressions are given in terms of the exponential integral functions of various order.

In Tables 1 and 2 results are shown for the criticality calculations carried out in slabs of varying thickness. They were determined from Eq. (3) utilizing a Cholesky-modified LR algorithm[11].

The algorithm makes the most of the positive definite, symmetric property of matrices, U and E. It is used here for the stability of computation and for saving the

Table 1. The Number of Secondary Neutrons per Slab Criticality
(The Linear Hermite Interpolation)

Slab Thickness	Exact	FEM Computation (% error)		
		N=2	N=4	N=8
11,3310	1.02	1.0216261 (0.1594)	1.0207239 (0.0710)	1.0207850 (0.0770)
6.6004	1.05	1.0522965 (0.2187)	1.0502420 (0.0230)	1.0500275 (0.0026)
4.2268	1.10	1.1025529 (0.2321)	1.1002386 (0.0217)	1.100186 (0.0017)
2.5786	1.20	1,2025393 (0.2116)	1.2002493 (0.0208)	1.2000402 (0.0033)
1.4732	1.40	1.4022732 (0.1624)	1.4002120 (0.0151)	1.4000238 (0.0017)
1.0240	1.60	1.6020478 (0.1280)	1.6001497 (0.0094)	1.5999741 (0.0016)
0.7776	1.80	1.8019241 (0.1069)	1.8001318 (0.0073)	1.7999639 (0.0020)
0.6220	2.00	2.0019863 (0.0993)	2.0002640 (0.0132)	2.0001010 (0.0050)

Tabele 2. The Number of Secondary Neutron per Collision for Slab Criticality
(The Cubic Hermite Interpolation)

Slab Thickness	Exact	FEM Computation (% error)		
		N=2	N=4	N=8
11.3310	1.02	1.0219963 (0.1957)	1.0193104 (0.0676)	1.0186892 (0.1285)
6.6004	1.05	1.0528780 (0.2741)	1.0474592 (0.2420)	1.0481820 (0.1731)
4.2268	1.10	1.1023345 (0.2122)	1.0962197 (0.3437)	1.0980651 (0.1759)
2.5785	1.20	1.2011160 (0.0558)	—	—
1.4732	1.40	—	—	—

calculational labour. The number of spatial mesh intervals is chosen 2, 4, and 8.

The comparision of the computed c with exact values shows that errors appear in the third to the fifth significant digit. Errors, especially in the case of the linear Hermite interpolation, decrease as the mesh interval becomes finer. It is noted that the cubic Hermite interpolation does not improve the accuracy which is obtained with the linear interpolation. This lack of accuracy seems accidental and is conjectured to be due to the numerical errors involved in the evaluation of matrix E.

Referring to Tables A-1 and A-2, the elements of E are combinations of exponental integral functions. These functions converge very slowly for arguments greater than 3~4. Therefore, it is likely that summation or difference of these functions at arguments close to one another may cancel significant figures and thus produce results of low accuracy. This conjecture is substantiated by observing the fact that a double precision computation of exponential integral functions results in errors less than a single precision calculation, by an order. Therefore, it seems that a more accurate evaluation of matrix E, though time-consuming, is required to determine c satisfactorily by the cubic Hermite interpolationg polynomials.

Table 3 lists the cubic results of the finite element solutions for the neutron flux near

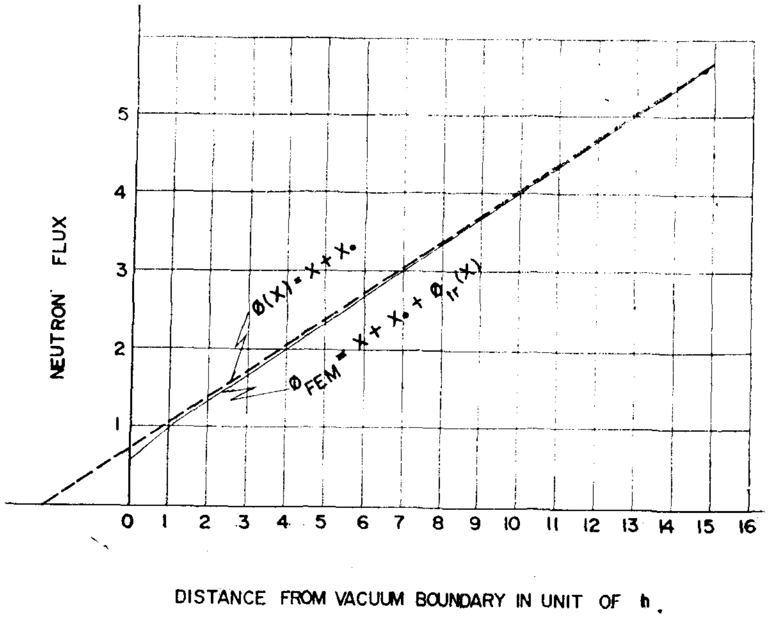

Fig. 2. Neutron Flux Near Vacnum Boundary

Table 3. Neutron Flux near Vacuum Boundary

Distance from Vacuum Boundary, n, in unit of $h=5/(N-1)$	Neutron Flux (Cubic Interpolation)		
	N=4	N=8	N=16
0	0.601658	0.588021	0.579574
1	2.362306	1.400005	0.997567
2	4.038839	2.130975	1.354195
3	5.706866	2.850159	1.698037
4		3.566009	2.036612
5		4.280921	2.372756
6		4.965572	2.707685
7		5.709929	3.041968
8			3.375888
9			3.709596
10			4.043177
11			4.376681
12			4.710135
13			5.043559
14			5.376961
15			5.710291

the vacuum boundary. Fig. 2 depicts them for the mesh number of $N=16$. According to Eq. (9), the flux determination involves an integration to infinity and an inversion of matrix $U-\frac{1}{2}E$ alike. The infinite integration is replaced by a finite one, assuming the neutron flux reaches the asymptotic value at the five mean free paths. As for the inversion of the matrix, a Cholesky decomposition technique[11] is adopted.

Table 4 compares the computed neutron flux with the exact solution in terms of 1 and x, which are defined by Eq's. (12) and (13), respectively. Unlike the case of the slab criticality, the cubic Hermite interpolation gives much better results than the linear interpolation. In the course of numerical computation, the diagonal elements of matrix U are observed to be more important to determine the neutron flux than the off-diagonal elements, whereas both the diagonal and the off-diagonal elements of E are equally important in the slab critic-

Table 4. Quantities L and x

Number of Basis Functions	L		x	
	Linear	Cubic	Linear	Cubic
4	1.415649	0.644026	0.690208	1.131641
8	0.731126	0.443262	0.615166	0.712738
16	0.518393	0.377282	0.537190	0.584046
32	0.4393388		0.511759	
Exact	0.358		0.546	

ality calculation. Therefore, it is conceived that any numerical errors made in the matrix E can affect the slab criticality more strongly than the flux determination. This may explain the reason why the cubic Hermite interpolation gives better results in flux computation than in the eigenvalue problem.

4. Conclusion

We obtained the finite element solutions to the slab criticality and the Milne problem in terms of the linear and the cubic Hermite interpolating polynomials. The lengthy integration is one formidable aspect of this method. Computation of the matrix E the elements of which are given by linear combinations of the exponential integral functions is most time-consuming.

The linear interpolation is found to give results which are in good agreement with the analytic solutions of the related transport equation. The cubic results agree well with the analytic solutions, yet are not so good as expected. This is ascribed to the way that we computed the matrix E. As pointed out in section 3, the exponential integral functions converge very slowly at arguments greater than 3~4. The evaluation of E from computing each of exponential integral funcitons individually may then result in a large cumulative error in the elements of the E matrix. Therefore, as one way to improve the cubic results, it is suggested that the linear combinations of exponential integrals are expanded in series[12] as a whole and the series is then used to compute the elements of the matrix E.

Acknowledgement

Authors are very grateful to Dr. Seong-Yun Kim and Mr. Si Hwan Kim for their helpful discussions on this problem. This work is supported by the Ministry of Science and Technology of the Republic of Korea government under the contract number R-76-56. The computations were performed at the Korea Institute of Science and Technology computer centers.

References

1) C.M. Kang and K.F. Hansen, Trans. Am. Nucl. Soc., 14, 199(1971); Nucl. Sci. Eng., 51, 456 (1973).
2) H.G. Kaper, C.K. Leaf, and A.J. Lindeman, ANL-7925, Argonne National Laboratory (1972).
3) L.A. Semenza, E.E. Lewis, and E.C. Rossow, Nucl. Sci. Eng., 47, 302 (1972).
4) L.O. Doppe and K.F. Hansen, Nucl. Sci. Eng., 54, 456 (1974).
5) T. Ohnishi, Procedings of a Seminar on Numerical Reactor Calculations, International Atomic Energy Agency, Vienna (1972).
6) W.F. Miller, Jr., E. E. Lewis, and E.C.

Rossow, Nucl. Sci. Eng., 51, 148 (1973).
7) E.E. Lewis, W.F. Miller, and T.P. Henry, Nucl. Sci. Eng., 58, 203 (1975).
8) G.I. Bell and S. Glasstone, *Nuclear Reactor Theory*, Van Nostrand Reinhold Co., New York (1970).
9) K.M. Case and P.F. Zweifel, *Linear Transport Theory*, Addison-Wesley Publishing Co., Reading, Mass. (1967).
10) I.K. Abu-Shumays and E.H. Bareis, Trans. Am. Nucl. Soc., 17, 236 (1973).
11) F.S. Acton, *Numerical Methods That Work*, Harper Row, New York (1970).
12) V.I. Pagurova, *Tables of Exponential Integral*, Pergamon Press, Oxford (1961).

Appendix Evaluation of of Matrix E

Case 1: The Linear Hemite Interpolating Polynomials

The Hermite interpolationg polynomials of linear type in equal intervals are defined as

$$U_n^{\circ}(x)=\begin{cases} u_n^{\circ -}=\frac{x}{h}+1-n & (n-1)h\le x\le nh \\ u_n^{\circ +}=n+1-\frac{x}{h} & nh\le x\le (n+1)h \\ 0 & \text{elsewhere} \end{cases} \quad (\text{A}-1)$$

h stands for the uniform mesh interval. It is straightforward to evaluate U using Equations (A-1) and (4). Computation of E is a little complicated. Since it is a symmetric matrix, it is sufficient to consider elements E_{ij} for $i\ge j$ Defining the columns and the rows of E in the order of $u_i^0(x)$; $i=0, 1, \cdots\cdots N$, one can show

$$\left.\begin{aligned} &E_{11}=I_3^{11} \\ &E_{i1}=I_2^{i1}+I_3^{i1} \\ &E_{ij}=I_1^{ij}+I_2^{ij}+I_3^{ij}+I_4^{ij} \end{aligned}\right\}\text{for } 1\le j\le i\le N$$

$$\left.\begin{aligned} &E_{N+1,\,1}=I_2^{N+1,\,1} \\ &E_{N+1,\,j}=I_1^{N+1,\,j}+I_2^{N+1,\,j} \\ &E_{N+1,\,N+1}=I_1^{N+1,\,N+1} \end{aligned}\right\}\text{for } 1< j\le N$$

where

$$I_1^{ij}=\int_{(i-2)h}^{(i-1)h} dx\int_{(j-2)h}^{(j-1)h} dx'\quad E_1(|x-x'|)\left(\frac{x}{h}+2-i\right)\left(\frac{x'}{h}+2-j\right)$$

$$=\begin{cases} \frac{1}{h^2}[-2E_5(h)-2h\ E_4(h)+\frac{1}{2}-\frac{1}{2}h^2+\frac{2}{3}h^3] \\ \qquad \text{for } i=j \\ \frac{1}{h^2}[-E_5(|i-j|h+h)-2E_5(|i-j|h)-E_5(|i-j|h-h)-hE_4(|i-j|h+h)+hE_4(|i-j|h-h)-h^2E_3(|i-j|h)] \quad \text{for } i\neq j \end{cases}$$

$$I_2^{ij}=\int_{(i-2)h}^{(i-1)h} dx\int_{(j-1)h}^{jh} dx'\quad E_1(|x-x'|)\left(\frac{x}{h}+2-i\right)\left(j-\frac{x'}{h}\right)$$

$$=\begin{cases} \frac{1}{h^2}[E_5((i-j)h)-2E_5((i-j-1)h)+E_5((i-j-2)h)+2hE_4((i-j)h)-2h\ E_4((i-j-1)h)+h^2E_3((i-j)h)] \quad \text{for } i-j\ge 2 \\ \frac{1}{h^2}[2E_5(h)+2hE_4(h)+h^2E_3(h)-\frac{1}{2}+\frac{1}{3}h^2] \\ \qquad \text{for } i-j=1 \\ \frac{1}{h^2}[E_5(2+j-i)h)-2E_5((1+j-i)h)+E_5((j-i)h)+2hE_4((1+j-i)h)-2hE_4((j-i)h)+h^2E_3((j-i)h] \quad \text{for } i-j\le 0 \end{cases}$$

$I_3^{ij}=I_1^{ji}$, and $I_4^{ij}=I_2^{ji}$.

Case 2: Cubic Hermite Interpolating Polynomials The cubic interpolating polynomials are given by

$$U_n^0(x)=\begin{cases} U_n^{0-}=3\left(\frac{x-(n-1)h}{h}\right)^2-2\left(\frac{x-(n-1)h}{h}\right)^3 \\ \qquad (n-1)h\le x\le nh \\ U_n^{0+}=3\left(\frac{(n+1)h-x}{h}\right)^2-2\left(\frac{(n+1)h-x}{h}\right)^3 \\ \qquad nh\le x\le (n+1)h \\ 0 \qquad \text{elsewhere} \end{cases}$$

$$U_n^1(x)=\begin{cases} U_n^{1-}=h\Big[-\Big(\dfrac{x-(n-1)h}{h}\Big)^2+\Big(\dfrac{x-(n-1)h}{h}\Big)^3\Big] & (n-1)h\leq x\leq nh \\ U_n^{1+}=h\Big[\Big(\dfrac{(n+1)h-x}{h}\Big)^2\Big]-\Big(\dfrac{(n+1)h-x}{h}\Big)^3\Big] & nh\leq x\leq (n+1)h \\ 0 & \text{elsewhere} \end{cases}$$

To define the columns and the rows of matrices E and U, we reorganized the labelling of the above element functions as

$$\left.\begin{aligned} f_{2i+1}(x)&=u_i^0(x)\\ f_{2i+2}(x)&=u_i^1(x)\end{aligned}\right\}\text{ for } i=1,2,\cdots\cdots N.$$

In terms of f functions, the elements of U and E are redefined by

$$U_{ij}=(f_i,f_j)=\int_o^a f_i(x)f_i(x)\ dx$$

$$E_{ij}=(f_i,Ef_j)=\int_o^a dxf_i\ (x)\int_o^a dx' E_1(|x-x'|)f_j(x')$$

U can be easily constructed referring to integrals in Ref. 1. It is laborious to construct E. After some manipulation, one can show that, for upper diagonal elements of E,

$$E_{11}=V(o),\quad E_{12}=D(o)$$

$$E_{1,\,2i+1}=V(-i)+\overline{V}(i),\ E_{1,\,2i+2}=-\overline{D}(i)+D(-i)\text{ for } i=1,\ \cdots\cdots,\ N-1.$$

$$E_{1,2N+1}=\overline{V}(N),\quad E_{1,2N+2}=-\overline{D}(N)$$

$$E_{22}=S(o)$$

$$\left.\begin{aligned} E_{2,\,2i+1}&=D(i)+\overline{D}(i)\\ E_{2,\,2i+2}&=S(i)-\overline{S}(i)\end{aligned}\right\}\text{ for } i=2,\cdots\cdots,\ N-1$$

$$E_{2,\,2N+1}=\overline{D}(N),\ E_{2,\,2N+2}=-\overline{S}(N)$$

$$\left.\begin{aligned} E_{2i+1,\,2j+1}&=2V(i-j)+\overline{V}(i-j)+\overline{V}(j-i)\\ E_{2i+1,\,2j+2}&=D(i-j)-\overline{D}(j-i)\\ E_{2i+1,\,2N+1}&=V(i-N)+\overline{V}(N-i)\end{aligned}\right\}\text{ for}$$

$$i\leq i\leq j\leq N$$

$$E_{2i+1,\,2N+2}=-D(N-i)-\overline{D}(N-i)$$

$$E_{2i+2,\,2j+2}=2S(i-j)-\overline{S}(i-j)-\overline{S}(j-i)$$

$$E_{2i+2,\,2j+1}=\overline{D}(j-i)-\overline{D}(i-j)$$

$$E_{2j+2,\ 2N+1}=\overline{D}(N-i)-D(i-N)$$

$$E_{2i+2,\ 2N+2}=S(N-i)-\overline{S}(N-i)$$

$$E_{2N+1,\,2N+1}=V(0),\quad E_{2N+1,\,2N+2}=-D(0),$$

$$E_{2N+2,\,2N+2}=S(0)$$

where

$$V(i)=\frac{9}{h^4}I^{22}\ (i)-6h^5\ I^{23}(i)-\frac{6}{h^5}I^{32}(i)+\frac{4}{h^6}I^{33}(i)$$

$$V(i)=\frac{9}{h_4}J^{22}(i-2)-\frac{12}{h^5}J^{23}\ (i-2)+\frac{4}{h^6}J^{33}\ (i-2)$$

$$D(i)=\frac{3}{h^3}I^{22}\ (i)-\frac{3}{h^4}I^{23}(i)\ -\frac{2}{h^4}I^{32}(i)+\frac{2}{h^5}I^{33}\ (i)$$

$$\overline{D}(i)=\frac{3}{h^3}J^{22}\ (i-2)-\frac{5}{h^4}J^{23}\ (i-2)+\frac{2}{h^5}J^{33}\ (i-2)$$

$$S(i)=\frac{1}{h^2}I_{22}\ (i)-\frac{1}{h^3}I^{23}(i)-\frac{1}{h^3}I^{32}(i)+\frac{1}{h^4}I^{33}\ (i)$$

$$\overline{S}(i)=\frac{1}{h^2}J^{22}\ (i-2)-\frac{2}{h^3}J^{23}\ (i-2)+\frac{1}{h^4}J^{33}\ (i-2)$$

$$I^{mn}(i)=\int_o^h dt\ t^m\int_o^h dt' t'^n E_1(|t-t'+ih'|)$$

$$J^{mn}(i)=\int_o^h dt\ t^m\int_o^h dt' t'^n E_1(|t+t'+ih|)$$

Table A-1 and A-2 listintegrals, $I^{mn}(i)$ and $J^{mn}(i)$.

Table A-1. Integrals I^{mn} (i) ($2\leq m,\ n\leq 3$)

	$i\geq 1$	$i=0$
$m=2$ $n=2$	$4[E_7((i+1)h-2E_7(ih)+E_7((i-1)h)]+4h[E_6(i+1)h)-E_6((i-1)h)]+2h^2[E_5((i+1)h)+E_5((i-1)h)]-h^4E_3(ih)$	$8E_7(h)+8h\ E_6(h)+4h^2E_5(h)-8E(o)+\frac{3}{4}h^3E_4(o)-h^4E_3(o)+\frac{2}{5}h^5E_2(o)$
$m=2$ $n=3$	$-12[E_8(i+1)h)-2E_8(ih)+E_8((i-1)h)]-12h[E_7((i+1)h)-E_7((i-1)h)]-6h^2[E_6((i+1)h)-E_6((i-1)h)]-2h^3[E_5(ih)-E_5((i-1)h)]+h^4E_4(ih)-h^5E_3(ih)$	$2h^3E_5(h)-2h^3E_5(o)+2h^4E_4(o)-h^5E_3(o)+\frac{1}{3}h^6E_2(o)$
$m=3$ $n=3$	$-36[E_9((i+1)h)-E_9(ih)+E_9((i-1)h)]-36h[E_8(i+1)h)-E_8((i-1)h)]-18h^2[E_7((i+1)h)+E_7((i-1)h)]-6h^3[E_6((i+1)h)-E_6((i-1)h)]-3h^4E_5(ih)-h^6E_3(ih)$	$-72E_9(h)-72h\ E_8(h)-36h^2E_7(h)-12h^3E_6(h)+72E_9(o)-3h^4E_5(o)-h^6E_3(o)+\frac{12}{5}h^6E_4(o)+\frac{2}{7}h^7E_2(o)$
$m=3$ $n=2$	$12[E_8((i+1)h)-2E_8(ih)+E_8((i-1)h)]+12h[E_7((i+1)h)-E_7((i-1)h)]+6h^2[E_6((i+1)h)+E_6((i-1)h)]+2h^3[E_5((i+1)h)-E_5(ih)]-h^4E_4(ih)-h^5E_3(ih)$	$I^{23}(o)$

Table A-2. Integrals J^{mn}(i) ($2\leq m,\ n\leq 3$)

	$i\geq 0$	$i=-1$	$i\leq -2$
$m=2$ $n=2$	$4[E_7((i+2)h)-2E_7((i+1)h)+E_7(ih)]+8h[E_6((i+2)h)-E_6((i+1)h)]+4h^2[2E_5((i+2)h)-E_5(i+1)h)]+4h^3E_4((i+2)h)+h^4E_3((i+2)h)$	$8E_7(h)+8hE_6(h)+8h^2E_5(h)+4h^3E_4(h)+h^4E_3(h)-8E_7(o)-4h^2E_5(o)+\frac{4}{3}h^3E_4(o)+\frac{1}{15}h^5E_2(o)$	$4[E_7(-ih)-2E_7((-i+1)h)+E_7)(-(i+2)h)]+8h[E_6(-(i+1)h-E_6(-(i+2)h)]-4h^2[E_5(-(i+1)h)+2E_5(-(i+2)h)]-4h^3E_4(-(i+2)h)+h^4E_3(-(i+2)h)$
$m=2$ $n=3$	$12[E_8(ih)-2E_8((i+1)h)+E_8((i+2)h)]-24h[E_7((i+1)h)-E_7((i+2)h)]12h^2[E_6((i+1)h)-2E_6((i+2)h)]-2h^3[E_5((i+1)h)-7E_5((i+2)h)]+5h^4E_4((i+2)h)+h^5E_3((i+2)h)$	$24hE_7(h)+24h^2E_6(h)+14h^3E_5(h)+5h^4E_4(h)+h^5E_3(h)-24hE_7(o)-2h^3E_5(o)+h^4E_4(o)+\frac{1}{30}h^6E_2(o)$	$-12[E_8(-ih)-2E_8(-(i+1)h)+E_8(-(i+2)h)]-24h[E_7(-(i+1)h-E_7(-(i+2)h)]+12h^2[E_6(-(i+1)h)-2E_6(-(i+2)h)]-2h^3[E_5(-(i+1)h)-7E_5(-(i+2)h)]-5h^4E_4(-(i+2)h)+h^5E_3(-(i+2)h)$
$m=3$ $n=2$	〃	〃	〃
$m=3$ $n=3$	$36[E_9(ih)-2E_9(i+1)h+E_9((i+2)h)]-72h[E_8((i+1)h)-E_8((i+2)h)]-36h^2[E_7((i+1)h)-2E_7((i+2)h)]-12h^3[E_6((i+1)h)-4E_6((i+2)h)]+21h^4E_5((i+2)h)+6h^5E_4((i+2)h)-h^6E_3((i+2)h)$	$72E_9(h)+72hE_8(h)+72h^2E_7(h)+48h^3E_6(h)+21h^4E_5(h)+6h^5E_4(h)+h^6E_3(h)-72E_9(o)-36h^2E_7(o)+\frac{3}{5}h^5E_4(o)+\frac{1}{70}h^7E_2(o)$	$36[E_9(-ih)-2E_9(-(i+1)h)+E_q(-(i+2)h)]+72[E_8(-(i+1)h)-E_8(-(i+2)h)]-36h^2[E_7(-(i+1)h)-2E_7(-(i+2)h)]+12h^3[E_6(-(i+1)h)-4E_6(-(i+2)h)]+21h^4E_5(-(i+2)h)-6h^5E_4((-i+2)h)+h^6E_3(-(i+2)h)$