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Ideal MHD Beta Limit for Optimum Stable Operation of Axisymmetric Tokamak Reactor with a Circular Cross Section

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원형 단면을 가진 축대칭형 토카막 핵융합로의 최적운전을 위한 이상적 자기유체역학 안전성을 유지하는 베타값의 최대한계

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Abstract

A method for determining the optimum ideal MHD β limit and the operation conditions is presented for an axisymmetric tokamak reactor with a circular cross section. The β limit is calculated under the constraints of ideal MHD instability criteria with varying the operation conditions which depend on the toroidal current density distributions. Semiempirical laws deduced from experimental observations are used for the toroidal current density distributions. Analytic derivations of various equations required to determine the β limit are carried out from the empirical equations. Various distributions of the β limit are obtained by the numerical calculations for different distributions of the toroidal current density. The resulting values of the maximum β limited by ideal MHD instabilities are expressed by a scaling law in terms of the tokamak geometry and the safety factor.

요 약

원형 단면을 가진 축대칭형 토카막 핵융합로에 적용할 수 있는 최적의 이상적 자기유체역학 베타값과 운선조건의 결정 방법을 제시하였다. 토로이달 전류밀도 분포로 조정되는 운전조건을 변화시키면서 이상적 자기유체역학 안정성을 유지시킬 수 있는 베타값의 한계를 계산하였다. 토로이달 전류밀도 분포에는 실험적 관찰들로부터 얻은 실험식들이 사용되었고, 베타값의 한계를 결정하기 위해 필요한 여러 식들이 이 실험식들로부터 유도되었다. 토로이달 전류밀도의 각각 다른 분포에 대해서 다양한 베타 한계값 분포들이 얻어졌다. 이상적 자기유체역학 불안정성들에 의해 제한받는 최대의 베타값을 토카막의 기하학적 변수와 안전인자에 의한 scaling law로 표현하였다.

1. Introduction

The maximum achievable β value is one of the most important parameters determining the relevance of the tokamak as an energy producing fusion device. β is a dimensionless form of plasma pressure indicating the efficiency of confining plasmas with magnetic field and is related to the thermal power of the fusion reactor. For example. the alpha particle heating power in JET fueled with 50%−50% DT, at the full field of 3.5T, can be approximated [1] by $P_\alpha \simeq \beta^2$, where β is the volume averaged beta in % and the power P_α is in megawatts. This means tokamak should have high β to be operated efficiently. There are, however, limitations on increasing the β value due to the MHD instabilities such as kink modes, internal modes, ballooning modes and so on. In particular, the limits set by the instabilities of ideal MHD provide a realistic guideline.

The fact that tokamak experiments have not yet reached β values as high as necessary lends great importance to the investigation of new configurations with higher β values. Many theoretical and experimental workers have tried to reach the highest β value against MHD instabilities[1-11]. Troyon and his coworkers have made an observation about the β limits in several tokamaks with neutral beam injection heatings [1,3,4], and Wesson and Sykes optimized the achievable β value with respect to ballooning modes alone[5].

It is the purpose of this work to determine the optimum toroidal current density distribution with the maximum achievable β value for the effective stable operation of the axisymmetric large-aspect-ratio tokamak with a circular cross section. With varying the shape of the current density profile based on semiempirical laws, the maximum achievable β value for each case is numerically obtained under the constraints of ideal MHD instability criteria. As a result the ultimate optimum β value and the corresponding current density profile are determined, and a scaling law for the maximum β limit is obtained in terms of system and operating parameters of the tokamak. The cgs electromagnetic units (e.m.u.) will be used throughout the present paper.

2. Beta Limitation and Stability Criteria

In an axisymmetric tokamak plasma with a circular cross section and a large aspect ratio ($R_0/a \gg 1$) the volume averaged β can be expressed as

$$\beta = \frac{16\pi}{a^2 B_\phi^2} \int_0^a Pr dr, \tag{1}$$

where a is the plasma radius and P is the plasma pressure. B_ϕ is the toroidal magnetic field which is assumed to be uniform in a large-aspect-ratio plasma region. To obtain the limit of B value the expression for B should include terms related to MHD instabilities. One of such parameters is α[5,12-14] which appears in the stability criterion of the ballooning mode and is defined as

$$\alpha \equiv -\frac{2R_0 q^2}{B_\phi^2} \frac{dP}{dr}, \tag{2}$$

in which another parameter q called the safety factor is included. α measures the pressure gradient of plasmas. By integration-by-part and using Eq · (2) the expression of β Eq · (1) becomes

$$\beta = \frac{4\pi}{a^2 R_0} \int_0^a \frac{\alpha r^2}{q^2} dr. \tag{3}$$

Since α depends on the ballooning mode criterion and q is related to the stability criterion of kink or internal mode, we can easily conjecture that β value is limited by the stability criteria through α and q. There are, therefore, two stability criteria which constrain β values. One is the Kruskal-Shafranov or Mercier criterion, and the other is the ballooning mode criterion.

The safety factor is a critically important quantity in the theory of MHD instabilities and appears in the Kruskal-Shafranov or Mercier criterion, which imposes a limit on the q-value in the plasma;

$$q_0 \geq 1 \tag{4}$$

at the magnetic axis of a large-aspect-ratio circular plasma. According to this stability criterion, tokamaks should be built with strong toroidal magnetic fields. Given the strongest toroidal field we can supply, the Kruskal-Shafranov or mercier criterion sets an upper bound on the amount of toroidal current we can drive in the plasma column.

The stability of ballooning modes driven by the interaction of the plasma pressure gradient with local regions of bad magnetic curvature is determined by minimizing the potential energy functional[7,12-14]. For a model problem representing a large-aspect-ratio tokamak with circular flux surfaces, when the magnetic field field is uniform over the magnetic surface but the shear is nonuniform, the Euler equation obtained by minimizing the potential energy functional becomes [12,13]

$$\frac{d}{d\eta}\,[1+(s\eta-\alpha\sin\eta)^2]\frac{dF}{d\eta}$$
$$+\alpha[\cos\eta+\sin\eta(s\eta-\alpha\sin\eta)]F=0,$$

wheres is the mean shear defined as

$$s(r)\equiv\frac{r}{q}\;\frac{dq}{dr} \qquad (6)$$

F is an eigenfunction related to the normal displacement of the perturbed fluid element in plasmas. η is related to the poloidal angle θ and its domain is $-\infty<\eta<\infty$. Derivation of E_q·(5) incorporates the Mercier criterion [12]. E_q · (5) has been integrated numerically with the boundary condition $F\rightarrow0$ as $|\eta|\rightarrow\infty$. The ballooning mode criterion from the numerical computations is, thus, well approximated by the following two equations. For small s and α the criterion becomes [13]

$$s=\gamma\alpha^2, \qquad (7)$$

where $\gamma=3/4(1+1/\sqrt{2})$, and for the other region of s and α, the approximation is [7]

$$4\alpha\left(1+\frac{5}{6}s\right)=1+2.8s^2. \qquad (8)$$

These two equations can be adopted as the stability criterion, but the regions of s and α where each of these equations is applicable, respectively, are not clear. In this work the intersecting point of E_{qs} · (7) and (8) is used as an inter-boundary of these two regions, which is found by the Newton's method;

$$s=7.323\times10^{-2}.$$

Hence the ballooning mode criterion becomes

$$\alpha=\sqrt{\frac{s}{r}},\quad 0\leqq s\leqq7.323\times10^{-2}\ 33;7,$$
$$\alpha=\frac{1+2.8s^2}{4(1+{}^5/_6s)},\quad s>7.323\times10^{-2}. \qquad (9)$$

Fig.1 shows the diagram of ballooning mode stability according to this criterion.

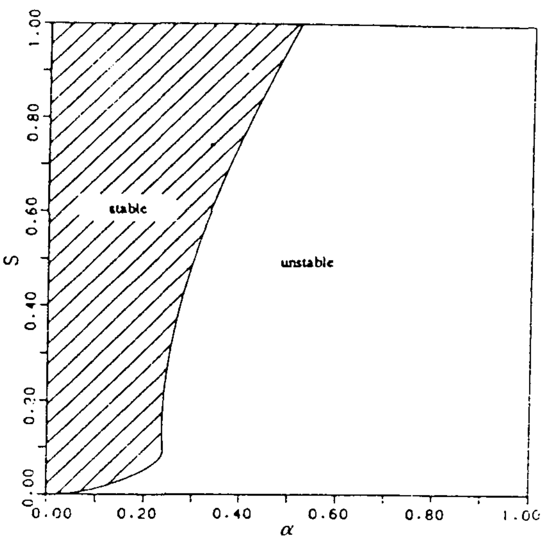


Fig.1 Stability Diagram of Balloning Mode.

3. Relation Between Stability Criteria Parameters and Toroidal Current Density

In view of the ballooning mode criterion (9) α is a function of s, and from Eq · $(6)_s$ depends on q. The safety factor for circular cross section is expressed as

$$q(r)=\frac{rB\phi}{R_0B_0(r)}, \qquad (9)$$

and in this case q has a relation with j_ϕ by the Ampere's law α is, therefore, dependent on j_ϕ. Consequently β is related to j_ϕ and its value depends on the shape of the toroidal current density profile. j_ϕ appears in the Grad-Shafranov equation as a source term. The Grad-Shafranov equation derived in the Grad-shafranov equation as a source trem. The Grad-Shafranov equation derived in the flux coordinate system (χ, ϕ, θ) for the axisymmetric plasma equilibria is [16]

$$R^2\nabla \cdot R^{-2}\nabla\chi = 4\pi R j_\phi = -4\pi R^2 \frac{dP(\chi)}{d\chi} - I(\chi)\frac{dI(\chi)}{d\chi}, \tag{10}$$

where χ is a flux surface label and $(\chi)I \equiv B_\phi R$ is a flux surface quantity related to total poloidal current. The two functions $P(\chi)$ and $I(\chi)$, which can be obtained accurately from transport codes, determine the toroidal current density,

$$j_\phi(R,\chi) = -R\frac{dP}{d\chi} - \frac{1}{8\pi R}\frac{dI^2}{d\chi}. \tag{11}$$

For simplicity without using transport codes, semiempirical laws deduced from experimental observations are used for the two functions P(x) and I(x) in this paper. Many empirical expressions of the toroidal current density have been used to solve plasma equilibrial [1,3,4,8-10,16]. In this work the functional forms in Ref.16 are used as follows;

$$P(\chi) = P_0\chi_S^{\nu},$$
$$I^2(\chi) = R_0^2[B_\phi^2(0) - B_\phi^2(a)]\chi_S^{\nu}, \tag{12}$$

where ν is a current peaking factor, P_0 is the plasma pressure at the magnetic axis, $B_\phi(0)$ and $B_\phi(a)$ are toroidal magnetic fields at the magnetic axis and at the plasma boundary, respectively. χ_S is a dimensionless poloidal magnetic flux normalized in such a way that

$$\chi_S = \frac{\chi_b - \chi}{\chi_b - \chi_o}, \tag{13}$$

where χ_o and χ_b are poloidal magnetic fluxes at the magnetic axis and at the boundary, respectively. A dimensionless parameter β_θ related to the poloidal beta β_p is defined as

$$\beta_\theta \equiv \frac{8\pi P_0}{B_\theta^2(a)} \tag{14}$$

where $B_\theta(a)$ is the poloidal magnetic field at the plasma boundary. Substitution of Eqs.(12) and (14) into (11) with the help of the pressure balance relation yields the current density

$$j_\phi(R,\chi) = -\frac{\nu}{2}\frac{\chi_S^{\nu-1}}{(\chi_o-\chi_b)} B_\theta^2(a) \frac{[R^2\beta_\theta + R_0^2(1-\beta_\theta)]}{4\pi R} \tag{15}$$

In Eq. (15) j_ϕ is a function of R and χ_S, which are not postion variables. In order to obtain the distribution of j_ϕ as a function of position variables, r and θ(ϕ is ignored due to the axisymmety of the torus), transformation of the coordinate system into a toroidal one (r,θ,ϕ) is carried out with an approximation of introducing the mean radius of the magnetic surface cross section. R is transformed into

$$R = R_0 + r\cos\theta. \tag{16}$$

A poloidal magnetic flux Ψ normalized to unity at the plasma surface is usually represented as [17,18] $\Psi = \omega^2$, where ω is the mean radius of the magnetic surface cross section and $\omega = 1$ at the plasma surface. But in our case, the poidal magnetic flux χ_S is normalized to unity at the magnetic axis. For making X_S normalized to unity at the plasma surface, let

$$\chi_S = 1-\omega^2 = 1-(\frac{r}{a})^2 \tag{17}$$

The conditions of Eqs. (13) and (17) at the magnetic axis and plasma boundary are identical. Inserting Eqs. (16) and (17) into (15) makes j_ϕ rewritten as

$$j_\phi(r,\theta) = \frac{j_{\phi 0}\nu}{R_0}[1-(\frac{r}{a})^2]^{\nu-1}\frac{[(R_0 + r\cos\theta)^2\beta_\theta + R_0^2(1-\beta_\theta)]}{R_0 + r\cos\theta} \tag{18}$$

where $j_{\phi 0}$ is the toroidal current density at the magnetic axis. By varying the values of ν and β_θ in Eq.(18) we can get various shapes of the toroidal current density distribution.

From the Ampere's law and Eq · (18), the poloidal magnetic field becomes

$$B_\theta(r)=\frac{j_{\phi o}{}^{\nu}}{r}\left\{\frac{\beta_\theta a^2}{2\nu}[1-(1-\frac{r^2}{a^2})^{\nu}]+R_0(1-\beta_\theta)f(\nu,r)\right\} \tag{19}$$

where $f(\nu,r)$ is defined as

$$f(\nu,r)\equiv\int_{\sqrt{R_0^2-r^2}}^{R_0}[1-\frac{1}{\varepsilon^2}+(\frac{t}{a})^2]^{\nu-1}dt,\ \varepsilon\equiv\frac{a}{R_0}. \tag{20}$$

The expression (19) of $B_\theta(r)$ yields the safety factor for circular cross section;

$$q(r)=\frac{q_o r^2}{2h(r)}, \tag{21}$$

where

$$q_o=\frac{2B_\phi}{j_{\phi o}R_o\nu}, \tag{22}$$

$$h(r)\equiv\left\{\frac{\beta_\theta a^2}{2\nu}[1-(1-\frac{r^2}{a^2})^{\nu}]+R_0(1-\beta_\theta)f(\nu,r)\right\}. \tag{23}$$

q_o should be greater than or equal to 1 to be stable against the internal or kink mode. By substitution of Eq · (21) into Eq · (6) and differentiation, the mean shear becomes

$$s(r)=2-\frac{rh_1(r)}{h(r)}, \tag{24}$$

where

$$h_1(r)\equiv r[1-\frac{r^2}{a^2}]^{\nu-1}\ [\beta_\theta+\frac{R_0(1-\beta_\theta)}{\sqrt{R_0^2-r^2}}]. \tag{25}$$

By varying the toroidal current density distribution (18) with the parameters of ν and β_θ, β limits are calculated from Eq · (3) with Eqs · (21) and (24) satisfying the constraints (4) and (9) of the stability criteria. The optimum stable operation condition is given for the toroidal current density distribution by choosing the appropreate values of ν and β_θ, and the maximum β value is then determined from these β limits.

4. Results and Discussions

1) Numerical Illustrations

Numerical calculations are carried out on the computer to find B limits and optimum current density profiles. The dependences of the profiles on the current peaking factor ν and the poloidal beta parameter β_θ are firstly examined for the toroidal current density in a large-aspect-ratio plasma ($r \langle\langle R_0$). Figs. 2 and 3

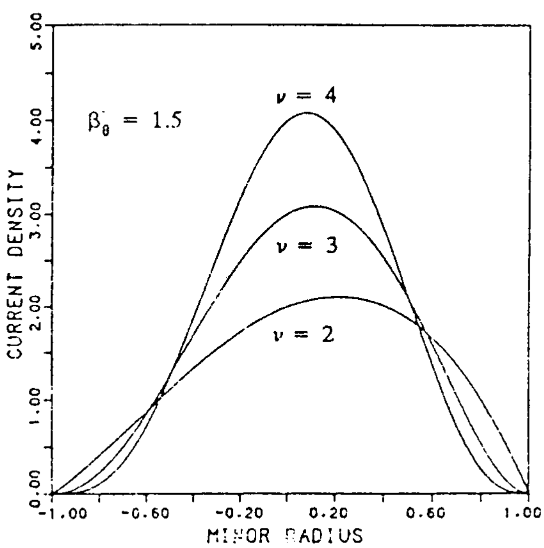


Fig.2 Profiles of Toroidal Current Density with Varying ν.

illustrate the toroidal current density profiles in an arbitrary unit with varying ν and β_θ, respectively. As ν increases the profile becomes centrally peaked shape (Fig 2), and as β_θ increases the peck shifts outward from the magnetic axis (Fig,3) while the area under the curve is not changed, that is, the total toroidal current is constant. In view of theseresults we are able to express in terms of ν and β_θ the arbitrary toroidal current density whose shape we know in the tokamak. Once we find the toroidal current density profile to give the maximum β value, we can take advantage of it as the optimum operating condition by controlling plasma generation and current drive in the tokamak.

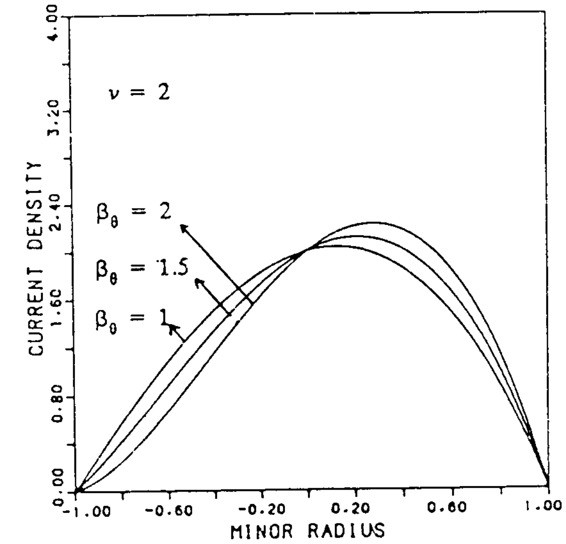
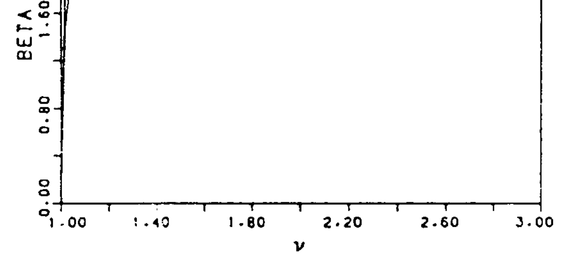


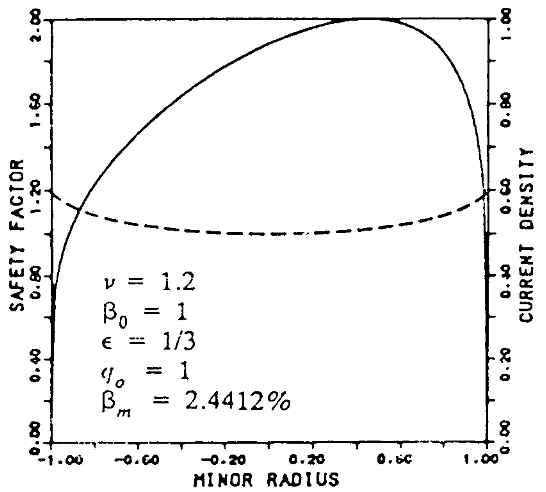
Fig.3 Profiles of Toroidal Current Density with Varying β_θ

Fig.4 shows β limits as functions of ν in three cases of β_θ1,1.5,2 for the inverse aspect ration $\varepsilon \equiv a/R_0 = 1/3$ and $q_0 = 1$. The maximum beta limit β_m of 2.65% is obtained when $\nu = 1.61$ and $\beta_\theta = 1$. As we have expected from the results of Fig.3, β limits have little



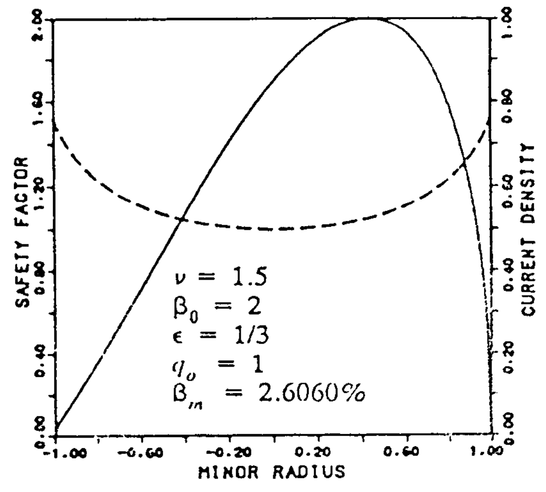
ig.4 Limit as a Function of ν.

dependence on β_θ. β_m is calculated to increase linearly as ε increases, which means that the fat type of tokamak reactor is good for high β operation. In our calculations, β_m is decreased as q_0 increases. Lower value of q_0 is them required for getting higher β_m, but q_0 is limited by kink or internal mode criterion, $q_0 \geqq 1$, and β_m is thus maximized when $q_0 = 1$. Three types of toroidal current density and safety factor profiles and their corresponding β limits are illustrated in Fig.5.a), b) andc) are characterized as flat, shifted and centrally-peaked types of toroidal current density profile, respectively. The centrally-peaked type is preferable rather than flat or shifted types to get the maximum β limit.



a) Flat Type,

Fig. 5 Three Types of Toroidal Current Density (Solid Lines) and Safety Factor (Dashed Lines) Profiles.



b) Shifted Type

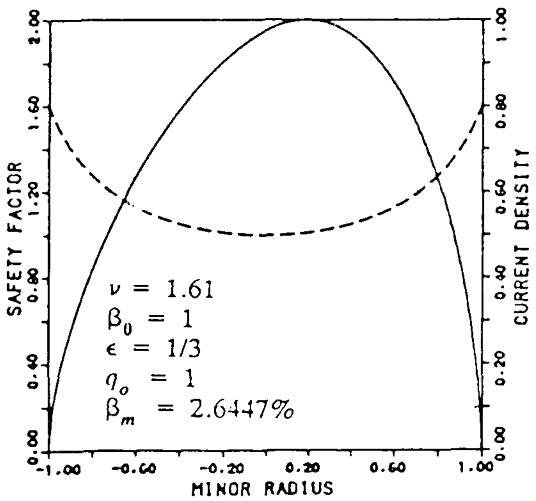


Fig. 5. c) Centrally-Peaked Type

2) Scaling Law

From the results the maximum beta limit β_m in is expressed in a scaling law when ν=1.61 as

$$\beta_m = 7.9\frac{\varepsilon}{q_0^2}, \tag{26}$$

which is the same form as that obtained from the two parameter model[2]. The difference is only the coefficient, which is 7.7 when adopting the two parameter model. Fig.6 shows the scaling law(26).

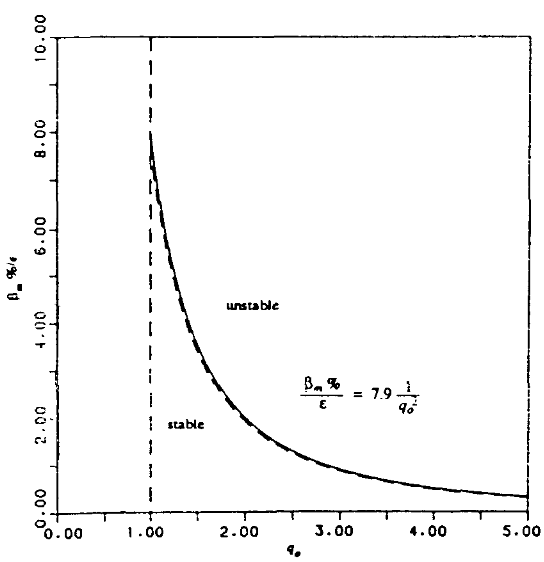


Fig. 6 Scaling Law for β.

- Solid Curve:Present Work (Semiempirical Formula) • Dashed Curve:Two Parameter Model
- Vertical Dashed Line:Mercier Criterion

5. Conclusion

Optimization of β limit is discussed with varying the shape of the toroidal current density profile for an axisymmetric tokamak with a circular cross section under the constraints of ideal MHD instabilities. Semiempirical laws are adopted for the toroidal current density distributions.

As the peaking factor ν increases the toroidal current density profile becomes peaked type, and when β_θ increases the peak of the profile shifts outward across the plasma column. β limit increases as the current density peak shifts to the magnetic axis, but the dependence of β limit on the peak shiftness is trivial. The maximum value of β limit is obtained when ν= 1.61 and β_θ=1, from which the optimum current density profile is determined for the effective stable operation of the tokamak. β_m in % is expressed by the scaling law,

$$\beta_m = 7.9\frac{\varepsilon}{q_0^2}.$$

In this work for high-temperature tokamak reactors ideal MHD instabilities are taken into account. A realistic β limit for ohmic-stage tokamaks may be obtained by including the resistive modes in the stability criteria due to low-temperature operation, and further study is required for these analyses. If ε=1/3 and q_0=1 we get $\beta_m \doteqdot 2.65\%$ according to the scaling law and this is nealy the upper limit for tokamaks with circular cross sections[11]. Noncircular shaping of the cross section of the plasma column is then recommended to increase the β limit.

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