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The Variational Method Applied to the Neutron Transport Equation

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Abstract

Noether's theorem is applied to the one dimensional neutron transport equation. It is obtained the transformation rendering the functional of the one dimensional Boltzmann equation invariant. It is derived the law conserving the product of the directional flux and its adjoint flux. The possible types of the solution of the Boltzmann equation are discussed. The results are compared with the well-known solution.

요　　약

Noether의 이론을 1차원의 중성자 수송방정식에 적용하였다. 1차원의 Boltzmann 방정식의 Functional을 불변케 하는 변환을 구했으며 이결과 중성자속과 그의 Adjoint 중성자속의 곱이 보존된다는 법칙을 유도하였다. 이 보존법칙으로부터 1차원의 Boltzmann방정식의 가능한 해의 형태를 얻었고 이것을 이미 알려진 해와 비교하였다.

I. Introduction

In recent years reactor physicists have used variational methods with great success[1]. Pomraning and Clark obtained an approximate solution of the monoenergetic Boltzmann equation by the trial function method[2]. Kaplan showed that there exists a close analogy between certain variational principles in reactor physics and these in classical mechanics[3]. Tavel, Clancy and Pomraning applied the Noether's theorem, one of the variational methods, to the diffusion equation, and they constructed an analogy between the diffusion equation and the classical mechanics[4]. They also showed an analogy between the diffusion equation and the time dependent Schroedinger equation.

In this paper, we would like to apply the Noether's theorem to the one dimensional Boltzmann transport equation. Basically, the fundamental idea of the Noether's theorem is to consider a variational characterization of a given equation, the corresponding functional of which is invariant under a continuous transformation of both the independent and dependent variables. Since the functional is invariant under the above transformation, its first variation vanishes. This fact can be used to derive certain relationships among the various variables of the problem under consideration. In the second section, following

Pomraning and Clark, we shall introduce a functional of the one dimensional Boltzmann transport equation. In the third section, a brief derivation of the Noether's theorem in the case of the functional including two independent and two dependent variables will be presented. In the following section, we will show how the theorem can be applied to the one dimensional Boltzmann equation and compare the result with the well-known solution. In the final section, we will discuss the results and offer several remarks.

II. Formalism

The equation of interest in reactor physics is the Boltzmann equation which can be given by

$$B\phi(x,\mu)=0, \tag{1}$$

where B is the one dimensional integro-differential Boltzmann operator. In the case of the constant cross-section, B is defined as

$$B\equiv\mu\frac{\partial}{\partial x}+1-\frac{c}{2}\int_{-1}^{1}d\mu'$$

Associated with the Boltzmann equation is the adjoint Boltzmann equation, written as

$$B^*\phi^*(x,\mu)=0, \tag{2}$$

where B^* is referred to as the adjoint Boltzmann operator and is defined as

$$\iint_{\Omega} dx d\mu\phi(x,\mu)\ B^*\phi^*(x,\mu) = \iint_{\Omega} dx d\mu\phi^*(x,\mu)B\phi(x,\mu) \tag{3}$$

The integrals in Eq. (3) extend over all of the phase space of interest.

The one dimensional monoenergetic Boltzmann equation with the constant cross-section is given by

$$\mu\frac{\partial\phi(x,\mu)}{\partial x}+\phi(x,\mu)-\frac{c}{2}\int_{-1}^{1}\phi(x,\mu')d\mu'=0 \tag{4}$$

By the defintion, the adjoint to Eq. (4) is easily shown to be

$$-\mu\frac{\partial\phi^*(x\ \mu)}{\partial x}+\phi^*(x,\ \mu)-\frac{c}{2}\int_{-1}^{1}\phi^*(x,\mu') \times d\mu'=0 \tag{5}$$

The boundary condition on the flux are(for the nonre-entrant problem under consideration)

$$\begin{aligned} &\phi\ (a,\mu)=0, && (0<\mu\leq 1)\\ &\phi\ (b,\mu)=0, && (-1\leq\mu<0)\\ &\phi^*(a,\mu)=0, && (-1<\mu\leq 0)\\ &\phi^*(b,\mu)=0, && (0<\mu\leq 1)\end{aligned}$$

where $a\leq x\leq b$.

Eq. (5) may then be written in terms of $\phi(x,\mu)$. If μ is replaced by $-\mu$ and $-\mu'$ by μ' in Eq. (5), we immediately see that

$$\phi^*(x,\mu)=\phi(x,-\mu). \tag{6}$$

Eq. (6) gives us a simple relation between the directional flux and its adjoint flux.

Let us consider the functional F describing this system. The functional F is given by Pomraning and Clark as follows

$$\begin{aligned} F[\phi,\ \phi^*] &= \iint_{\Omega} dx d\mu\phi^*\ B\phi\\ &= \iint_{\Omega} dx d\mu\ \phi\ B^*\phi^*. \end{aligned}$$

By equating the first variation of F with respect to ϕ and ϕ^* equal to zero, we can get Eq. (4) and Eq. (5). Thus rendering this functional stationary is equivalent to solving both the Boltzmann equation and its adjoint equation.

III. Noether's Theorem

Let us introduce two dimensional Noether's theorem[5]. Following Tavel, Clancy and Pomraning[4], we seek the first variation of the integral

$$F\equiv\iint_{\Omega} L[x,\mu,\phi(x,\mu),\phi^*(x,\mu),\phi_x(x,\mu), \phi_\mu(x,\mu),\phi_x^*(x,\mu),\phi_\mu^*(x,\mu)]dxd\mu, \tag{7}$$

when $\phi(x,\mu)$, $\phi^*(x,\mu)$ and the limits of integration are allowed to vary. We consider a family of transformations depending on a parameter ε,

$$x^+(x,\mu,\phi(x,\mu),\ \phi^*(x,\mu);\varepsilon)\equiv x^+(x,\mu;\varepsilon), \tag{8}$$

$$\mu^+(x,\mu,\phi(x,\mu),\phi^*(x,\mu);\varepsilon)\equiv\mu^+(x,\mu;\varepsilon), \tag{9}$$

$$\psi^+(x,\mu,\psi(x,\mu),\psi^*(x,\mu);\varepsilon)\equiv\psi^+(x,\mu;\varepsilon), \tag{10}$$

$$\psi^{*+}(x,\mu,\psi(x,\mu),\psi^*(x,\mu);\varepsilon)\equiv\psi^{*+}(x,\mu;\varepsilon), \tag{11}$$

These transformations are assumed one-to-one continuously differentiable (with respect to ε) and further are assumed to reduce to the identity transformations for $\varepsilon=0$.

The integral F in the transformed variables is written as

$$F(\varepsilon)=\iint_{\Omega^+} L[x^+,\mu^+,\psi^+(x^+\mu^+;\varepsilon),\psi^{*+}(x^+,\mu^+;\varepsilon)\ \psi^+_{x^+}(x^+,\mu^+;\varepsilon),\ \psi^{*+}_{x^+}(x^+,\mu^+;\varepsilon),\ \psi^+_{\mu^+}(x^+,\mu^+;\varepsilon).\ \psi^{*+}_{\mu^+}(x^+,\mu^+;\varepsilon)]\times dx^+d\mu^+, \tag{12}$$

where the region of integration Ω^+ maps on to the original region Ω by the transformations Eqs. (8) and (9). Making a change of integration variable in Eq. (12) from Ω^+ to Ω, we have

$$F(\varepsilon)=\iint_{\Omega} L[x^+,\mu^+,\psi^+(x^+,\mu^+;\varepsilon),\ \psi^{*+}(x^+,\mu^+;\varepsilon),\ \psi^+_{x^+}(x^+,\mu^+;\varepsilon),\ \psi^{*+}_{x^+}(x^+,\mu^+;\varepsilon),\ \psi^+_{\mu^+}(x^+,\mu^+;\varepsilon),\ \psi^{*+}_{\mu^+}(x^+,\mu^+;\varepsilon)]\frac{\partial(x^+,\mu^+)}{\partial(x,\mu)}dxd\mu \tag{13}$$

The first variation of Eq. (13) is formed by differentiation with respect to ε. Since the limits of integration are independent of ε, the $\partial/\partial\varepsilon$ operator can be taken inside the integral. We find

$$\delta F=\iint_{\Omega}[L_x\delta x+L_\mu\delta_\mu+L\psi\delta\psi+L\psi^*\delta\psi^*+L\psi_x\delta(\psi_x)+L\psi_\mu\delta(\psi_\mu)+L_{\psi_x}{}^*\delta(\psi)^*_x+L_{\psi_\mu}{}^*\delta(\psi_\mu{}^*)+L(\delta x)_x+L(\delta\mu)_\mu]dxd\mu \tag{14}$$

where we have introduce the variation notation

$$\delta F\equiv\varepsilon\left(\frac{\partial F(\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{15}$$

$$\delta\psi\equiv\varepsilon\left(\frac{\partial\psi^+(x^+,\mu^+;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{16}$$

$$\delta\psi^*\equiv\varepsilon\left(\frac{\partial\psi^{*+}(x^+,\mu^+;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{17}$$

$$\delta(\psi_x)\equiv\varepsilon\left(\frac{\partial\psi^+_{x^+}(x,{}^+\mu^+;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{18}$$

$$\delta(\psi_\mu)\equiv\varepsilon\left(\frac{\partial\psi^+_{\mu^+}(x^+,\mu^+;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{19}$$

$$\delta(\psi_x{}^{*+})\equiv\varepsilon\left(\frac{\partial\psi^{*+}_{x^+}(x^+,\mu^+;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{20}$$

$$\delta(\psi_\mu{}^{*+})\equiv\varepsilon\left(\frac{\partial\psi^{*+}_{\mu^+}(x^+,\mu^+;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{21}$$

$$\delta x\equiv\varepsilon\left(\frac{\partial x^+(x,\mu;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{22}$$

$$\delta\mu=\varepsilon\left(\frac{\partial\mu^+(x,\mu;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{23}$$

In Eqs. (16) through (23), the partial derivatives with respect to ε are taken with x and μ (not x^+ and μ^+) held constant. As indicated by Tavel et al[4). in the one dimensional case, at this point one would like to write $\delta(\psi_x)=(\delta\psi)_x$, $\delta(\psi_\mu)=(\delta\psi)_\mu$, $\delta(\psi_x{}^*)=(\delta\psi^*)$ and $\delta(\psi_\mu{}^*)=(\delta\psi^*)_\mu$ in Eq. (14) and perform integrations by parts. However, the processes of differentiations with respect to x and μ do not commute with computing the variation, i. e. $\delta(\psi_x)\neq(\delta\psi)_x$, $(\delta\psi_\mu)\neq(\delta\psi)_\mu$, $\delta(\psi_x{}^*)\neq(\delta\psi^*)_x$ and $\delta(\psi_\mu{}^*)\neq(\delta\psi^*)_\mu$. The reason that these two operators do not commute(in most variational calculations they do) is that an ε-dependent transformation has been applied to the independent as well as the dependent variable (in most variational calculations $\delta x=0$ and $\delta\mu=0$). Because of this commutation problem, we introduce a second type of variation $\bar{\delta}\psi$ and $\bar{\delta}\psi^*$ defined as

$$\bar{\delta}\psi\equiv\varepsilon\left(\frac{\partial\psi^+(x,\mu;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{24}$$

$$\bar{\delta}\psi^*\equiv\varepsilon\left(\frac{\partial\psi^{*+}(x,\mu;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0}, \tag{25}$$

Comparison of Eqs. (16) and (17), and Eqs. (24) and (25) shows that the difference between $\bar{\delta}\psi$ and $\bar{\delta}\psi^*$, and $\delta\psi$ and $\delta\psi^*$ is that $\bar{\delta}\psi$ and $\bar{\delta}\psi^*$ have fixed argument points, x and μ, whereas the argument points x^+ and μ^+ for $\delta\psi$ and $\delta\psi^*$ depend on ε. Use of the chain rule establishes the identity

$$\delta\psi=\bar{\delta}\psi+\psi_x\delta x+\psi_\mu\delta\mu, \quad (26)$$

$$\delta\psi^*=\bar{\delta}\psi^*+\psi_x{}^*\delta x+\psi_\mu^*\delta\mu \quad (27)$$

Similarly, we define

$$\bar{\delta}(\psi_x)\equiv\varepsilon\left(\frac{\partial\psi_x{}^+(x,\mu,\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0} \quad (28)$$

$$\bar{\delta}(\psi_\mu)\equiv\varepsilon\left(\frac{\partial\psi_\mu{}^+(x,\mu;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0} \quad (29)$$

$$\bar{\delta}(\psi_x{}^*)\equiv\varepsilon\left(\frac{\partial\psi_x{}^{*+}(x,\mu;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0} \quad (30)$$

$$\bar{\delta}(\psi_\mu{}^*)\equiv\varepsilon\left(\frac{\delta(\psi_\mu{}^{*+}(x,\mu;\varepsilon)}{\partial\varepsilon}\right)_{\varepsilon=0} \quad (31)$$

The barred variations are connected with the unbarred ones as follows

$$\delta(\psi_x)=\bar{\delta}(\psi_x)+\psi_{xx}\delta x+\psi_{x\mu}\delta\mu, \quad (32)$$

$$\delta(\psi_\mu)=\bar{\delta}(\psi_\mu)+\psi_{\mu x}\delta x+\psi_{\mu\mu}\delta\mu, \quad (33)$$

$$\delta(\psi_x{}^*)=(\bar{\delta}\psi_x{}^*)+\psi_{xx}{}^*\delta x+\psi_{x\mu}{}^*\delta\mu, \quad (34)$$

$$\delta(\psi_\mu{}^*)=\bar{\delta}(\psi_\mu{}^*)+\psi_{\mu x}{}^*\delta x+\psi_{\mu\mu}{}^*\delta\mu \quad (35)$$

We have the useful commutation property

$$\bar{\delta}(\psi_x)=(\bar{\delta}\psi)_x, \quad (36)$$

$$\bar{\delta}(\psi_\mu)=(\bar{\delta}\psi)_\mu, \quad (37)$$

$$\bar{\delta}(\psi_x{}^*)=(\bar{\delta}\psi^*)_x, \quad (38)$$

$$\bar{\delta}(\psi_\mu{}^*)=(\bar{\delta}\psi^*)_\mu, \quad (39)$$

Using Eqs. (24) through (39) and following Courant and Hilbert,[5] we obtain

$$\begin{aligned}\delta F=&\iint_\Omega[L_x\delta x+L_\mu\delta_\mu+L_\psi\delta\psi+L_{\psi}{}^*\delta\psi^*+L_{\psi_x}\delta(\psi_x)\\&+L_{\psi_\mu}\delta(\psi_\mu)+L_{\psi_x{}^*}\delta(\psi_x{}^*)+L_{\psi_\mu{}^*}\delta(\psi_\mu{}^*)\\&+L(\delta x)_x+L(\delta\mu)_\mu]dxd\mu\\=&\iint_\Omega\Big\{\Big[\frac{\partial L}{\partial\psi}-\frac{\partial}{\partial x}\frac{\partial L}{\partial\psi_x}-\frac{\partial}{\partial\mu}\frac{\partial L}{\partial\psi_\mu}\Big]\bar{\delta}\psi\\&+\Big[\frac{\partial L}{\partial\psi^*}-\frac{\partial}{\partial x}\frac{\partial L}{\partial\psi_x{}^*}-\frac{\partial}{\partial\mu}\frac{\partial L}{\partial\psi^*}\Big]\bar{\delta}\psi^*\\&+\frac{\partial}{\partial x}(L\delta x+L_{\psi_x}\bar{\delta}\psi+L_{\psi_x}{}^*\bar{\delta}\psi^*)\\&+\frac{\partial}{\partial\mu}(L\delta\mu+L_{\psi_\mu}\bar{\delta}\psi+L_{\psi_\mu}{}^*\bar{\delta}\psi^*)\Big\}dxd\mu\end{aligned} \quad (40)$$

We now suppose that the transformations given by Eqs. (8), (9), (10) and (11) are such that the first variation of F is zero for an arbitrary region of integration. Then Eq. (40) yields

$$\begin{aligned}\delta F=&\iint_\Omega\Big\{\Big[\frac{\partial L}{\partial\psi}-\frac{\partial}{\partial x}\frac{\partial L}{\partial\psi_x}-\frac{\partial}{\partial\mu}\frac{\partial L}{\partial\psi_\mu}\Big]\bar{\delta}\psi\\&+\Big[\frac{\partial L}{\partial\psi^*}-\frac{\partial}{\partial x}\frac{\partial L}{\partial\psi_x{}^*}-\frac{\partial}{\partial\mu}\frac{\partial L}{\partial\psi_\mu{}^*}\Big]\bar{\delta}\psi^*\\&+\frac{\partial}{\partial x}(L\delta x+L_{\psi_x}\bar{\delta}\psi+L_{\psi_x}{}^*\bar{\delta}\psi^*)\\&+\frac{\partial}{\partial\mu}(L\delta\mu+L_{\psi_\mu}\bar{\delta}\psi+L_{\psi_\mu}{}^*\bar{\delta}\psi^*)\Big\}dxd\mu=0\end{aligned} \quad (41)$$

Since Eq. (41) holds for arbitrary region Ω, the integrand must vanish; i.e.,

$$\begin{aligned}&\Big[\frac{\partial L}{\partial\psi}-\frac{\partial}{\partial x}\frac{\partial L}{\partial\psi_x}-\frac{\partial}{\partial\mu}\frac{\partial L}{\partial\psi_\mu}\Big]\bar{\delta}\psi\\&+\Big[\frac{\partial L}{\partial\psi^*}-\frac{\partial}{\partial x}\frac{\partial L}{\partial\psi_x{}^*}-\frac{\partial}{\partial\mu}\frac{\partial L}{\partial\psi_\mu{}^*}\Big]\bar{\delta}\psi^*\\&+\frac{\partial}{\partial x}(L\delta x+L_{\psi_x}\bar{\delta}\psi+L_{\psi_x}{}^*\bar{\delta}\psi_*)\\&+\frac{\partial}{\partial\mu}(L\delta\mu+L_{\psi_\mu}\bar{\delta}\psi+L_{\psi_\mu}{}^*\bar{\delta}\psi^*)=0.\end{aligned} \quad (42)$$

If the function L is the Lagrangian for an equation of interest, Eq. (42) becomes

$$\begin{aligned}&\frac{\partial}{\partial x}(L\delta x+L_{\psi_x}\bar{\delta}\psi+L_{\psi_x}{}^*\bar{\delta}\psi^*)\\&+\frac{\partial}{\partial\mu}(L\delta\mu+L_{\psi_\mu}\bar{\delta}\psi+L_{\psi_\mu}{}^*\bar{\delta}\psi^*)=0\end{aligned} \quad (43)$$

because, in this case, it holds

$$\frac{\partial L}{\partial\psi}-\frac{\partial}{\partial x}\frac{\partial L}{\partial\psi_x}-\frac{\partial}{\partial\mu}\frac{\partial L}{\partial\psi_\mu}=0, \quad (44)$$

$$\frac{\partial L}{\partial\psi^*}-\frac{\partial}{\partial x}\frac{\partial L}{\partial\psi_x{}^*}-\frac{\partial}{\partial\mu}\frac{\partial L}{\partial\psi_\mu{}^*}=0 \quad (45)$$

Eq. (43) is Noether's theorem for the two dimensional case (two independent and two dependent variables).

VI. Application to the Boltzmann Equation

The Monoenergetic Boltzmann transport equation in the slab geometry is given by

$$B\psi=\mu\frac{\partial\psi(x,\mu)}{\partial x}+\psi(x,\mu)-\frac{c}{2}\int_{-1}^{1}\psi(x,\mu')d\mu'=0 \quad (46)$$

and the associated functional F,

$$F=\iint\psi^*B\psi dxd\mu=\iint\Big[\mu\psi^*\frac{\partial\psi(x,\mu)}{\partial x}+\psi^*\psi-\frac{c}{2}\psi^*\int_{-1}^{1}\psi(x,\mu')d\mu'\Big]dxd\mu. \quad (47)$$

Functional F should be invariant by certain transformations. Now we consider the transformations

$$x^+(x,\mu,\psi,\psi^*;\varepsilon)=x, \quad (48)$$

$$\mu^{+}(x,\mu,\phi,\phi^{*};\varepsilon)=\mu \tag{49}$$

$$\phi^{+}(x,\mu,\phi,\phi^{*};\varepsilon)=\phi\exp(\varepsilon a), \tag{50}$$

$$\phi^{*+}(x,\mu,\phi,\phi^{*};\varepsilon)=\phi^{*}\exp(-\varepsilon a), \tag{51}$$

where a is a constant. These particular transformations obviously leave the functional F invariant. From Eqs. (22) and (23), we have

$$\delta x=0, \tag{52}$$

$$\delta\mu=0. \tag{53}$$

From Eqs. (26), (27), (29), (30), (52) and we have

$$\bar{\delta}\phi=\delta\phi-\phi_x\delta x-\phi_\mu\delta\mu=\delta\phi=\varepsilon a\phi, \tag{54}$$

$$\bar{\delta}\phi^{*}=\delta\phi^{*}-\phi_x{}^{*}\delta x-\phi_\mu{}^{*}\delta\mu=\delta\phi^{*}=-\varepsilon a\phi^{*}. \tag{55}$$

Thus Noether's theorem in our case yields

$$\frac{\partial}{\partial x}(L\delta x+L_{\phi_x}\delta\phi+L_{\phi_x^*}\delta\phi^{*})+\frac{\partial}{\partial\mu}(L\delta\mu+L_{\phi_\mu}\delta\phi+L_{\phi_\mu^*}\delta\phi^{*})=0, \tag{56}$$

where $L\equiv\mu\phi^{*}\frac{\partial\phi}{\partial x}+\phi^{*}\phi-\frac{c}{2}\phi^{*}\int_{-1}^{1}\phi(x,\mu)d\mu'$.

By the definition of L, we obtain

$$L_{\phi_x}\equiv\frac{\partial L}{\partial\phi_x}=\mu\phi^{*}, \tag{57}$$

$$L_{\phi_x^*}=L_{\phi_\mu}=L_{\phi_\mu^*}=0. \tag{58}$$

Substituting Eqs. (52), (53), (54), (55), (57) and (58) into Eq. (56), we have

$$\frac{\partial}{\partial x}(\varepsilon a\mu\phi\phi^{*})=0. \tag{59}$$

From this, it is noted that $\phi\phi^{*}$ is independent of the space variables x; i.e., $\phi\phi^{*}$ is conserved. Noting that $\phi^{*}(x,\mu)=\phi(x,-\mu)$, we have

$$\phi(x,\mu)\phi^{*}(x,\mu)=\phi(x,\mu)\phi(x,-\mu)\equiv G(\mu) \tag{60}$$

Since the left hand side is an even function of μ, $G(\mu)$ is also an even function. From Eq. (46), it can be inferred that the solution of the Boltzmann equation should be of the types.

1. $\phi(x,\mu)=\phi(\mu)e^{-f(\mu)g(x)}$,
   where $f(\mu)$ is an odd function.
2. $\phi_\nu(x,\mu)=\phi_\nu(\mu)e^{-\nu g(x)}$,
   where ν is chosen to satisfy
   $\phi_\nu(x,-\mu)=\phi_{-\nu}(-\mu)e^{+\nu g(x)}$

The second type of the solution shows that $\phi(x,\mu)$ can be solved by the method of separation of variables. Putting $\phi_\nu=\phi_\nu(\mu)e^{-\nu g(x)}$ into Eq. (46), we see that $g(x)$ should be equal to x. This result is in good agreement with the well-known solution.[6] A more detailed calculation of this type is given in "Linear Transport Theory" by Case and Zweifel.[6]

V. Remarks

In this work, our main purpose has been to apply the Noether's theorem to the one dimensional Boltzmann equation. We have treated the problem in the case of constant cross-section and source free. Thus we have found the fact that the product of the directional flux and its adjoint flux is conserved. The generalization of our case to the practical one would be more interesting. If any other transformation rendering the functional invariant be sought, the corresponding new result would be derived. It should be noted that we can get an approximate solution by substituting the first type of solution into the original equation.

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