Element Free Lattice Boltzmann Method for Fluid-Flow Problems

Jong Chull Jo[a*], Kyung Wan Roh[a], Young Gill Yune[a], Hho Jhung Kim[a], and Young Kwon[b]
a *Safety Issue Research Dept., Korea Institute of Nuclear Safety, 19 Kusung-dong, Yusung-gu, Daejon 305-338, Korea, *jcjo@kins.re.kr*
b US Naval Postgraduate School

1. Introduction

The Lattice Boltzmann Method (LBM) has been developed for application to thermal-fluid problems. Most of the those studies considered a regular shape of lattice or mesh like square and cubic grids. In order to apply the LBM to more practical cases, it is necessary to be able to solve complex or irregular shapes of problem domains. Some techniques were based on the finite element method [1, 2].

Generally, the finite element method is very powerful for solving two- or three-dimensional complex or irregular shapes of domains using the iso-parametric element formulation which is based on a mathematical mapping from a regular shape of element in an imaginary domain to a more general and irregular shape of element in the physical domain. In addition, the element free technique is also quite useful to analyze a complex shape of domain because there is no need to divide a domain by a compatible finite element mesh.

This paper presents a new finite element and element free formulations for the lattice Boltzmann equation using the general weighted residual technique. Then, a series of validation examples are presented.

2. Finite Element LBM Formulation (FELBM)

The lattice Boltzmann equation is expressed as

$$\frac{\partial f_\alpha}{\partial t} + \vec{e}_\alpha \cdot \nabla f_\alpha = \Omega_\alpha \qquad (1)$$

where f_i is the particle velocity distribution function along the α direction, t represents time, $\vec{e}_\alpha$ is the discrete velocity vector along the α direction, and Ω_α denotes the collision operator.

When a single relaxation time technique is used for the collision operator like the BGK technique [3], the collision operator can be written as

$$\Omega_\alpha = -\frac{1}{\tau}\left(f_\alpha - \tilde{f}_\alpha\right) \qquad (2)$$

where τ is the relaxation constant and $\tilde{f}_\alpha$ denotes the local equilibrium distribution of f_α. The local equilibrium distribution is

$$\tilde{f}_\alpha = \rho\omega_\alpha\left[1 + \frac{3\vec{v}\cdot\vec{e}_\alpha}{c^2} + \frac{9\left(\vec{v}\cdot\vec{e}_\alpha\right)^2}{2c^4} - \frac{3\vec{v}\cdot\vec{v}}{2c^2}\right] \qquad (3)$$

in which ρ is the fluid density and $\vec{v}$ is the fluid velocity. They can be expressed as

$$\rho = \sum_\alpha f_\alpha, \qquad \rho\vec{v} = \sum_\alpha f_\alpha \vec{e}_\alpha \qquad (4)$$

In addition, ω_α for D2Q9 and for D3Q15 (see Fig. 1) are the weighting parameter for each velocity direction as given below, respectively:

$$\omega_\alpha = \begin{pmatrix} 4/9 & \alpha = 0 \\ 1/9 & \alpha = 1,2,3,4 \\ 1/36 & \alpha = 5,6,7,8 \end{pmatrix}, \qquad \omega_\alpha = \begin{pmatrix} 2/9 & \alpha = 0 \\ 1/9 & \alpha = 1 \text{ to } 6 \\ 1/72 & \alpha = 7 \text{ to } 14 \end{pmatrix} \qquad (5)$$

Substitution of Eq. (3) into Eq. (1) results in

$$\frac{\partial f_\alpha}{\partial t} + \vec{e}_\alpha \cdot \nabla f_\alpha + \frac{1}{\tau}\left(f_\alpha - \tilde{f}_\alpha\right) = 0 \qquad (6)$$

In order to derive the Finite Element Lattice Boltzmann Method (FELBM) from Eq. (6), the problem domain is discretized into a number of finite elements. Then, the variable f_α is expressed in terms of the interpolation functions and nodal variables as given below:

$$f_\alpha = \sum_{i=1}^{n} H^i f_\alpha^i = [H]\{f_\alpha\} \qquad (7)$$

in which H^i is the spatial interpolation function for the nodal variable f_α^i at the $i-th$ node of the finite element, and n is the number of nodes per element. In addition, $[H]$ is a row vector consisting of the interpolation functions, and $\{f_\alpha\}$ is a column vector containing unknown solutions at the nodes. Plugging Eq. (7) into Eq. (6) yields

$$[H]\{\dot{f}_\alpha\} + \vec{e}_\alpha \cdot [\nabla H]\{f_\alpha\} + \frac{1}{\tau}[H]\left(\{f_\alpha\} - \{\tilde{f}_\alpha\}\right) = 0 \qquad (8)$$

for each finite element. The superimposed dot denotes the temporal derivative. Applying the weighted residual formulation to Eq. (8) gives the following expression

$$\sum\int_{S_e}\{w\}\left([H]\{\dot{f}_\alpha\} + \vec{e}_\alpha \cdot [\nabla H]\{f_\alpha\} + \frac{1}{\tau}[H]\left(\{f_\alpha\} - \{\tilde{f}_\alpha\}\right)\right)dS = 0 \qquad (9)$$

where the integration is conducted over each finite element domain S_e and the summation is performed over the total number of elements. Furthermore, $\{w\}$ is a column vector of the weighting functions. The size of $\{w\}$ is equal to the number of nodes per element. Rewriting Eq. (9) yields

$$[M]\{\dot{F}_\alpha\} + [K]\{F_\alpha\} + [C]\{F_\alpha\} - [C]\{\tilde{F}_\alpha\} = 0 \qquad (10)$$

where

$$[M] = \sum[m] = \sum\int_{S_e}\{w\}[H]dS, \quad [K] = \sum[k] = \sum\int_{S_e}\{w\}(\vec{e}_\alpha \cdot [\nabla H])dS$$

$$[C] = \sum[c] = \sum\int_{S_e}\frac{1}{\tau}\{w\}[H]dS, \quad \{F_\alpha\} = \sum\{f_\alpha\}, \quad \{\tilde{F}_\alpha\} = \sum\{\tilde{f}_\alpha\}$$

Depending on the choice of the weighting functions, the subsequent technique can be called the Galerkin method, collocation method, method of moments, least-square method, or sub-domain method. In this study, the first three techniques will be presented.

Using the forward difference scheme for time integration, Eq. (10) is expressed as

$$\{F_\alpha\}^{t+\Delta t} = \{F_\alpha\}^t + \Delta t[M]^{-1}\left([C]\{\tilde{F}_\alpha\}^t - [C]\{F_\alpha\}^t - [K]\{F_\alpha\}^t\right) \qquad (11)$$

Equation (11) is solved for the given initial and boundary conditions.

3. Element Free LBM Formulation (EFLBM)

Consider a lattice point x and its neighborhood as a subdomain Ω_x. Inside the subdomain, there are n numbers of randomly located lattice points. In order to represent the solution $u(x)$ inside the subdomain, a polynomial expression is assumed as below

$$u(\boldsymbol{x}) = \{p(\boldsymbol{x})\}^T\{a(\boldsymbol{x})\} \qquad (12)$$

where $\{p(x)\}$ is a vector containing a complete monomial basis of order m as expressed below and $\{\alpha(x)\}$ is a vector consisting of coefficients of the monomial terms. The coefficient vector is determined to best fit the solutions at the lattice points inside the subdomain, utilizing the

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weighted least square technique. The sum of the weighted least square is expressed as

$$J(\boldsymbol{x}) = \sum_{k=1}^{n} w_k(\boldsymbol{x}) \left[\{p(\boldsymbol{x}_k)\}^T \{a(\boldsymbol{x})\} - \hat{u}_k \right]^2 \tag{13}$$

where $w_k(x)$ is the weighting function associated with the lattice point k, and $\hat{u}_k$ is the solution at the same node. Minimization of the above equation with respect to $\{\alpha(x)\}$ results in

$$[A]\{\alpha\}=[B]\{\hat{u}\} \tag{14}$$

$$[A(\boldsymbol{x})] = \sum_{k=1}^{n} w_k(\boldsymbol{x}) \{p(\boldsymbol{x}_k)\}\{p(\boldsymbol{x}_k)\}^T$$

$$[B(\boldsymbol{x})] = \left[w_1(\boldsymbol{x})\{p(\boldsymbol{x}_1)\} \quad w_2(\boldsymbol{x})\{p(\boldsymbol{x}_2)\} \quad \ldots \quad w_n(\boldsymbol{x})\{p(\boldsymbol{x}_n)\} \right] \tag{15}$$

Solving for the coefficient vector $\{\alpha(x)\}$ and substituting the resulting expression into Eq. (12) gives

$$u(x)=\{\Phi\}^T\{\hat{u}\} \tag{16}$$

where the interpolation function vector $\{\Phi\}$ is shown as

$$\{\Phi(\boldsymbol{x})\}^T = \{p(\boldsymbol{x})\}^T [A(\boldsymbol{x})]^{-1} [B(\boldsymbol{x})] \tag{17}$$

The interpolation function, Eq. (17), is applied the weighted residual formulation as described in the previous section in order to develop the Element-Free Lattice Boltzmann Method (EFLBM).

This interpolation function vector is different from that used in the finite element method. For the latter, the interpolation function vector satisfies the following relationship

$$u(x_k)=\hat{u}_k \tag{18}$$

For the derivative of $[A(x)]^{-1}$, the following expression is used [4].

$$\left([A]^{-1}\right)_l = -[A]^{-1}([A])_l[A]^{-1} \tag{19}$$

4. Numerical Results and Discussion

The first example was a two-dimensional steady-state Poiseuille flow between two parallel walls. The EFLBM was applied to solve the problem and compared in Fig. 2. In order not to make the figures too crowded, separate figures were plotted for each comparison. Both EFLBM and FELBM solutions agreed very well with the analytical solution. In this as well as subsequent figures, unless otherwise mentioned, all velocities were normalized with respect to the maximum velocity value. In addition, the distance was normalized in terms of the wall spacing.

The second example was a two-dimensional unsteady Couette flow between two parallel plates with spacing h, one of which was stationary while the other began to move at a constant velocity U. The EFLBM and FELBM solutions are compared to the analytical solution as a function of time in Fig. 2. They all agreed very well.

The last example was a cavity driven flow. The top side was subject to a contact velocity u and the other sides were rigid walls. The Reynolds number ud/ν was considered to be 100, where d is the dimension of a square cavity, and ν is the fluid viscosity. The present FELBM solution was compared to the solution in Ref. [5]. Figure 3 shows the horizontal velocity profile along the vertical centerline of the domain and the vertical velocity profile across the horizontal centerline of the domain. Both velocities agreed very well between the two solutions.

5. Conclusions

The FELBM and EFLBM were developed from the lattice Boltzmann equation by applying the weighted residual formulations. These techniques allow users to model any complex shape of domain in 2-D or 3-D. As a result, theses techniques will be used for the fluid-structure interaction analysis code. Especially, FELBM will be extended further for subsequent work because of its maturity and compatibility with structural finite element models.

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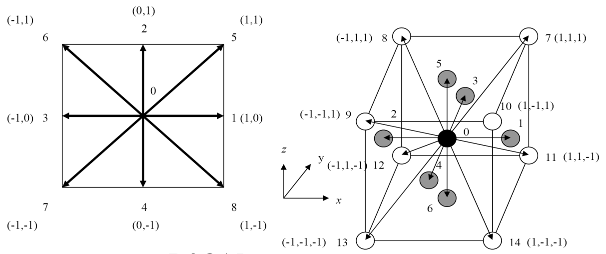

Fig.1 D2Q9 and D3Q15 lattices showing discrete 9 and 15 velocity vectors, respectively.

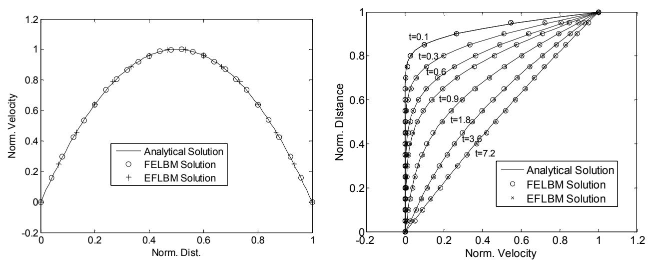

Fig. 2 Plots of the normalized velocity profiles for Plane Poiseuille (left) and unsteady Couette (right) flows.

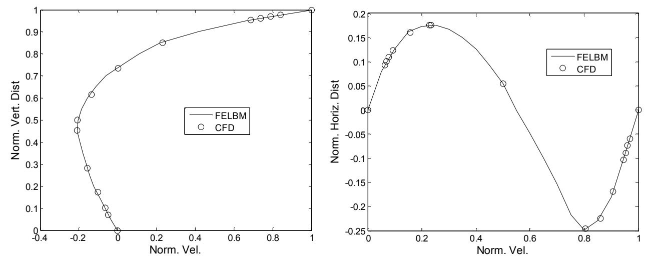

Fig. 3. Horizontal (left) and vertical (right) velocity profiles along each centerline of the domain.