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Preliminary Formulation of Finite Element Solution for the 1-D, 1-G Time Dependent Neutron Diffusion Equation without Consideration about Delay Neutron

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1. Introduction

The most attractive feature of the Finite Element Method (FEM) is geometrical freedom. If time-dependent equation is solved with the FEM, the limitation of the input geometry will disappear. It has often been pointed out that the numerical methods implemented in the RFSP code are not state-of-the-art. Although an acceleration method such as the Coarse Mesh Finite Difference (CMFD) for Finite Difference Method (FDM) does not exist for the FEM, one should keep in mind that the number of time steps for the transient simulation is not large. The rigorous formulation in this study will richen the theoretical basis of the FEM and lead to an extension of the dynamics code to deal with a more complicated problem. In this study, the formulation for the 1-D, 1-G Time Dependent Neutron Diffusion Equation (TDNDE) without consideration of the delay neutron will first be done. A problem including one multiplying medium will be solved. Also several conclusions from a comparison between the numerical and analytic solutions, a comparison between solutions with various element orders, and a comparison between solutions with different time differencing will be made to be certain about the formulation and FEM solution.

2. Formulation

2.1 TDNDE with Constant Properties

If the material properties do not change over time and the Fick's law is applied, TDNDE can then be written as:

$$\frac{1}{v}\frac{\partial\phi(x,t)}{\partial t}=v\Sigma_f\phi(x,t)-\left\{-D\frac{\partial^2\phi(x,t)}{\partial x^2}+\Sigma_a\phi(x,t)\right\}$$

where v, $v\Sigma_f$, D, Σ_a and $\phi(x,t)$ denote the neutron speed (cm/sec), macroscopic nu-fission cross section (/cm), diffusion coefficient (cm), macroscopic absorption cross section (/cm) and neutron flux, respectively.

2.2 Applying Weighted Residual Method and Galerkin Method

The weighted residual method is used to make the weighted residual zero for an arbitrary weighting function for whole domain [1]. Using the integration by parts, the following equation is satisfied:

$$\frac{1}{v}\int_\Omega\frac{\partial\phi(x,t)}{\partial t}w(x)dx$$
$$+\left\{D\int_\Omega\frac{\partial\phi(x,t)}{\partial x}\frac{\partial w(x)}{\partial x}dx-D\frac{\partial\phi(x,t)}{\partial x}w(x)\Big|_{LE}^{RE}+\left(\Sigma_a-v\Sigma_f\right)\int_\Omega\phi(x,t)w(x)dx\right\}=0$$

where the $w(x)$ and Ω (cm) are an arbitrary weighting function and entire domain, respectively.

In this step, the Galerkin method provides us with the form of and approximated solution and a weighting function as follows:

$$\phi(x,t)=\sum_{j=1}^{N}\phi^j(t)u^j(x)$$
$$w^i(x)=u^i(x),(i=1,N)$$

where N, $\phi^k(t)$, $u^k(x)$, $w^k(x)$ denote the number of nodes, flux amplitude at the k-th node, basis function at the k-th node and the k-th weighting function.

Using the above equations and albedo expressions for the currents, the following equation can be established:

$$\left[\sum_{j=1}^{N}\int_\Omega\frac{1}{v}u^i(x)u^j(x)dx\frac{d\phi^j(t)}{dt}\right]$$
$$+\left[\sum_{j=1}^{N}\left\{\begin{aligned}&\int_\Omega D\frac{du^i(x)}{dx}\frac{du^j(x)}{dx}+\left(\Sigma_a-v\Sigma_f\right)u^i(x)u^j(x)dx\\&+\alpha_{RE}u^i(RE)u^j(RE)+\alpha_{LE}u^i(LE)u^j(LE)\end{aligned}\right\}\phi^j(t)\right]=0$$

where α_{RE} and α_{LE} denote the albedos at the right and left ends.

2.3 Matrix Form and theta method for time discritization

The so-called capacitance (or mass) matrix and stiffness matrix for N number of weighting functions [2] can be defined as follows:

$$\sum_{j=1}^{N}\boldsymbol{C}_{ij}\frac{d\phi^j(t)}{vdt}+\sum_{j=1}^{N}\boldsymbol{K}_{ij}\phi^j(t)=0\quad,\quad 0\le t\le T,\ 1\le i\le N$$

where,

$$\boldsymbol{C}_{ij}=\int_\Omega u^i(x)u^j(x)dx$$
$$\boldsymbol{K}_{ij}=\int_\Omega D\frac{du^i(x)}{dx}\frac{du^j(x)}{dx}+\left(\Sigma_a-v\Sigma_f\right)u^i(x)u^j(x)dx+\alpha_{RE}u^i(RE)u^j(RE)+\alpha_{LE}u^i(LE)u^j(LE)$$

Using the theta method and an assumption of a constant time step size, the following equation is satisfied:

$$\left(\mathbf{C}+v\Delta t\theta\mathbf{K}\right)\boldsymbol{\phi}^{n+1}=\left(\mathbf{C}-v\Delta t(1-\theta)\mathbf{K}\right)\boldsymbol{\phi}^{n}$$

where Δt (sec) and θ are the time step size and a constant respectively.

2.4 Element Mapping

Each component of the capacitance and stiffness matrix includes integration for the entire domain. This integration is same as the summation of an individual integration of the element. Also, every integrations for an element can be transposed using a local coordinate (or length coordinate in 1-D):

$$\boldsymbol{C}_{ij} = \sum_{h=1}^{N_E} \int_{\hat{\Omega}} \hat{u}_h^{ok(i,h)}(\xi) \hat{u}_h^{ol(j,h)}(\xi) \left| J_h^o \right| d\xi$$

$$\boldsymbol{K}_{ij} = \sum_{h=1}^{N_E} \int_{\hat{\Omega}} D \left(\frac{du_h^{ok(i,h)}(\xi)}{d\xi} \frac{d\xi^o}{dx} \right) \left(\frac{du_h^{ol(i,h)}(\xi)}{d\xi} \frac{d\xi^o}{dx} \right) \left| J_h^o \right| d\xi$$

$$+\alpha_{RE} u_h^{ok(i,h)}(1) u_h^{ol(i,h)}(1) + \alpha_{LE} u_h^{ok(i,h)}(-1) u_h^{ol(i,h)}(-1) + \left(\Sigma_a - \nu \Sigma_f \right) \boldsymbol{C}_{ij}$$

$$\left| J_h^o \right| = \sum_{i=1}^{N_o+1} x_{hi} \frac{d\hat{u}_h^{oi}(\xi)}{d\xi}$$

where N_E, $k(i,h)$, $\hat{u}_h^{ok(i,h)}$, $\left| J_h^o \right|$ and N_o are the number of elements, local node index k for the i-th global node in the h-th element, the k-th basis function in local coordinate in the h-th element of order o, determinant of jacoby matrix in the h-th element of order o, and the element order, respectively.

3. Numerical Result

Because of the homogeneity of the problem, an analytic solution can be easily calculated. A total of 12 cases are tested, and the period for each case is calculated by a linear extrapolation. The initial flux is assumed that constant over the domain.

Table I: Problem Description

D	Σ_a	$\nu\Sigma_f$	v
1.2	0.12	0.125	6000
Ω	N^E	T	Δt
100	10	0.0999	0.0001

Table II: Input Cases Description

$\alpha_{LE}=0.0$, $\alpha_{RE}=0.0$			$\alpha_{LE}=0.5$, $\alpha_{RE}=0.5$		
	$\theta=0.0$	$\theta=0.5$		$\theta=0.0$	$\theta=0.5$
$N_o=1$	Case 1	Case 4	$N_o=1$	Case 7	Case 10
$N_o=2$	Case 2	Case 5	$N_o=2$	Case 8	Case 11
$N_o=3$	Case 3	Case 6	$N_o=3$	Case 9	Case 12

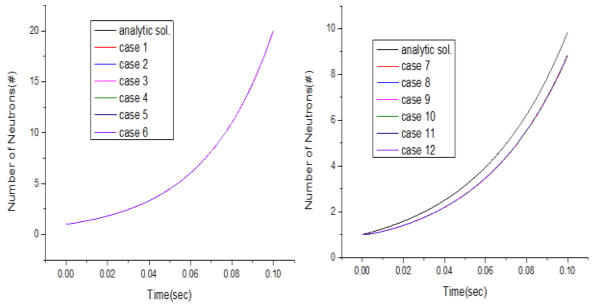

Fig. 1. Power Plots for Reflective B.C. and Vacuum B.C.

Table III: Periods for Cases

Analytic Sol.			Analytic Sol.		
0.0333			0.0436		
Case 1	Case 2	Case 3	Case 7	Case 8	Case 9
0.0334	0.0334	0.0334	0.0495	0.0494	0.0494
Case 4	Case 5	Case 6	Case 10	Case 11	Case 12
0.0333	0.0333	0.0333	0.0495	0.0493	0.0943

For case 7 through case 12, the period prediction is not good because a flat initial flux condition is used instead of a steady state calculation. The numerical results using the initial flux in a sine shape are also calculated.

Table IV: Periods for Cases using Sine Shape Initial Flux

Case 7	Case 8	Case 9	Case 10	Case 11	Case 12
0.0418	0.0418	0.0418	0.0418	0.0417	0.0417

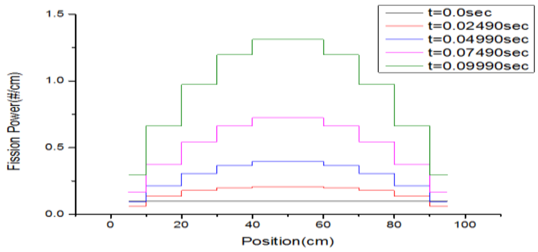

Fig. 2. Fission Power of Case 12 with Time for a Flat Initial Flux

4. Conclusions

By investigating various cases with different values of albedo, theta, and the order of elements, it can be concluded that the finite element solution is agree well with the analytic solution. The higher the element order used, the higher the accuracy improvements are obtained. The Crank-Nicolson Method($\theta=0.5$) is better than the Explicit Euler Method($\theta=0.0$) in this problem. For the case in which the boundary condition is not reflective, the flux shape influences the leakage and that it is necessary to find the eigenvector by calculating the steady state-calculation. An extension to multi-group, multi-dimension, heterogeneous problems, and including delay neutron will be done in the near future.

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