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Two Adjoint Solutions for the Analytic Function Expansion Nodal Method in the Hexagonal Geometry

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Abstract

The methods for obtaining the mathematical and the physical adjoint solutions to the neutron diffusion equation formulated with the equivalence theory in the hexagonal geometry are presented under the framework of the AFEN-type nodal methods. The mathematical adjoint coupling equations whose coefficients have the opposite meanings in a response matrix sense to those of the forward coupling equations are derived by transposing the forward nodal coupling equations. Although it has been believed that it is impossible or nearly impossible to derive the physical adjoint coupling equations when discontinuity factors are involved in nodal methods, the physical adjoint coupling equations are derived based on the adjoint current discontinuity across interfaces instead of the flux discontinuity in the forward coupling equations. Two adjoint fluxes have turned out to be identical in the case of finite difference formulation with discontinuity factors. The results of a numerical test show that the physical adjoint flux defined here is consistent to the mathematical adjoint flux even though discontinuity factors are involved.

1 Introduction

The perturbation theory provides a means to predict the response of dependent parameters (e.g., reactivity or reaction rates) to relatively small perturbations in independent parameters (e.g., minor changes in the core loading pattern, feedback parameters, or control rod positions) without actually calculating the perturbed configuration, but rather by using the unperturbed quantities. The perturbation formula generally requires not only the forward solution but also the adjoint solution. For the use of perturbation theory in the reactor analysis employing a nodal method, the adjoint solution scheme must be established for the specific nodal method.

Lawrence (Lawrence, 1984) introduced two different adjoint solutions in his discussion on the perturbation theory for the Nodal Expansion Method (NEM). *The physical adjoint flux* is the solution $\mathbf{f}_1^*$ of the matrix equation $\mathbf{L}^* \mathbf{f}_1^* = \mathbf{0}$ where $\mathbf{L}^*$ is the internodal coupling matrix obtained by discretizing the adjoint differential diffusion equation. On the other hand, *the mathematical adjoint flux* is the solution $\mathbf{f}_2^*$ of the matrix equation $\mathbf{L}^T \mathbf{f}_2^* = \mathbf{0}$ where $\mathbf{L}^T$ is the transpose of the coupling matrix associated with the forward nodal unknowns, which are discretized from the forward diffusion equation. If $\mathbf{L}^*$ is equal to $\mathbf{L}^T$ as in the Finite Difference Method (FDM), the two adjoint fluxes become identical. $\mathbf{L}^*$ and $\mathbf{L}^T$ are not generally identical for higher order nodal methods.

Cacuci criticized the terminology introduced by Lawrence and reminded us of the work done by him and two others (Cacuci, 1997) where two adjoints are referred to as *the operator formalism* and *the matrix formalism*. In spite of its inappropriateness, we keep Lawrence's terminology throughout this paper due to its familiarity to the researchers in the field of nodal methods.

Considering the responses to be predicted by the perturbation theory are certainly the derived quantities from the solution of the forward equations at the perturbed state which is already discretized by a nodal method, we can imagine that the mathematical adjoint is required in the perturbation theory. Therefore, obtaining the physical adjoint solution is only meaningful when the physical adjoint is much easier to obtain than the mathematical adjoint and it is so close to that of the mathematical adjoint that it might be used as an approximation of the mathematical adjoint.

If the discontinuity factors introduced by the equivalence theory are not involved in a nodal method, the physical adjoint flux is relatively easier to obtain than the mathematical adjoint flux. Since the shapes of two adjoint fluxes have turned out to be very close for most nodal methods, the physical adjoint flux is often used as a good approximation. But it has been generally perceived that, when the equivalence theory is employed to allow the flux to be discontinuous across the node boundary, it is impossible or very difficult to define the physical adjoint flux. In this work, although we are still not sure of its good agreement to the mathematical adjoint flux and its simplicity in obtaining, we successfully define the physical adjoint flux by allowing the current, instead of the flux to be discontinuous across the node boundary.

The perturbation formula contains the inner product of gradients of the forward flux and the adjoint flux. It has been recognized (Lawrence, 1984) that it would be inaccurate to use only the node-average forward and adjoint fluxes in approximating the inner product. The accuracy can be maintained only when all the forward unknowns adopted in a specific nodal method and their adjoint partners are included in the perturbation formula. As indicated in the numerical test result section, the considerable mismatches between the two adjoint fluxes at corner points are noted if the corner-point fluxes are adopted as nodal unknowns in the nodal method. Some mathematical adjoint fluxes at corner points often become negative with very small absolute values, while the physical adjoint fluxes at the same corner-points are still positive with the magnitude of node-average adjoint fluxes. Therefore, it should be noted the use of the physical adjoint flux as an approximation of the mathematical adjoint flux may result in relatively large errors in perturbation calculations by employing such a nodal method. Moreover, since we can use the same algorithm to solve the forward flux and the mathematical adjoint flux, there is no reason to practically use the physical adjoint flux instead of the mathematical adjoint flux. This comes from the fact that transposing even the very complicated forward coupling equations involving other variables such as the transverse leakages in addition to nodal fluxes can never become impossible and results in the same structure of coupling equations for the mathematical adjoint flux for most cases.

In this work, we developed the mathematical and the physical adjoint flux models for the Analytic Function Expansion Nodal (AFEN) method (Noh, 1994; Cho, 1995A) and its variations such as the Polynomial Expansion Nodal (PEN) method and the AFEN/PEN hybrid nodal method (Cho, 1995C; Noh, 1996) with the general equivalence theory in the hexagonal geometry. The adjoint flux for the rectangular geometry NEM was first studied by Lawrence (Lawrence, 1984) and followed by Kim and Kim (Kim, 1996), and those for the hexagonal geometry NEM by Yang, Taiwo, and Khalil (Yang, 1993). Cho and Hong (Cho, 1995B) have developed the adjoint flux model for the AFEN method without the discontinuity factors in the rectangular geometry.

2 Adjoint Solutions of the AFEN-Type Nodal Methods

2.1 Forward Equations

The diffusion equation for a hexagonal node *n* shown in Figure 1 is given by

$$-\mathbf{D}^n \hat{\mathrm{f}}_n(x,y) + \mathrm{S}^n \hat{\mathrm{f}}_n(x,y) = \frac{1}{k_{eff}} \mathrm{cnS}_f^n \hat{\mathrm{f}}_n(x,y), \tag{1}$$

where the hat ^ on the neutron flux denotes the homogeneous flux, which is allowed to be discontinuous across node interfaces or corner points.

The detail derivation of the nodal coupling equations for the AFEN-type nodal methods are described in the references : Noh (1994) and Cho (1995A). The resultant coupling equations consist of three different types of equations: the nodal neutron balance equations to be solved for the node-average fluxes, the interface current continuity equations for the interface fluxes, and the corner-point balance (CPB) equations for the corner-point fluxes. Here, we retrieve the expressions for an interface current and for a corner-point leakage which provides the foundations for all coupling equations:

$$\begin{aligned}\tilde{\mathbf{J}}_{x0}^n &= \left(\mathbf{a}_1^n + \mathbf{a}_2^n\right)\left(\tilde{\mathbf{F}}_{x0}^n\right)^{-1}\tilde{\mathrm{f}}_{x0}^n + \left(\mathbf{a}_1^n - \mathbf{a}_2^n\right)\left(\tilde{\mathbf{F}}_{x1}^n\right)^{-1}\tilde{\mathrm{f}}_{x1}^n + \left(\mathbf{a}_3^n + \mathbf{a}_4^n\right)\frac{1}{2}\left\{\left(\tilde{\mathbf{F}}_{u1}^n\right)^{-1}\tilde{\mathrm{f}}_{u1}^n + \left(\tilde{\mathbf{F}}_{p1}^n\right)^{-1}\tilde{\mathrm{f}}_{p1}^n\right\} \\ &+ \left(\mathbf{a}_3^n - \mathbf{a}_4^n\right)\frac{1}{2}\left\{\left(\tilde{\mathbf{F}}_{u0}^n\right)^{-1}\tilde{\mathrm{f}}_{u0}^n + \left(\tilde{\mathbf{F}}_{p0}^n\right)^{-1}\tilde{\mathrm{f}}_{p0}^n\right\} + \left(\mathbf{b}_1^n + \mathbf{b}_2^n\right)\frac{1}{2}\left\{\left(\mathbf{F}_{u0}^n\right)^{-1}\mathrm{f}_{u0}^n + \left(\mathbf{F}_{p1}^n\right)^{-1}\mathrm{f}_{p1}^n\right\} \\ &+ \left(\mathbf{b}_1^n - \mathbf{b}_2^n\right)\frac{1}{2}\left\{\left(\mathbf{F}_{u1}^n\right)^{-1}\mathrm{f}_{u1}^n + \left(\mathbf{F}_{p0}^n\right)^{-1}\mathrm{f}_{p0}^n\right\} + \mathbf{b}_3^n\left\{\left(\mathbf{F}_{u0}^n\right)^{-1}\mathrm{f}_{u0}^n + \left(\mathbf{F}_{p1}^n\right)^{-1}\mathrm{f}_{p1}^n\right\} - 2\left(\mathbf{a}_1^n + \mathbf{a}_3^n + \mathbf{b}_1^n + \mathbf{b}_3^n\right)\bar{\mathrm{f}}_n,\end{aligned} \tag{2}$$

$$\begin{aligned}\mathbf{L}_{x0}^n &= \left(\mathbf{b}_4^n + \mathbf{b}_5^n\right)\left(\mathbf{F}_{x0}^n\right)^{-1}\mathrm{f}_{x0}^n + \left(\mathbf{b}_4^n - \mathbf{b}_5^n\right)\left(\mathbf{F}_{x1}^n\right)^{-1}\mathrm{f}_{x1}^n + \left(\mathbf{b}_6^n + \mathbf{b}_7^n\right)\frac{1}{2}\left\{\left(\mathbf{F}_{u1}^n\right)^{-1}\mathrm{f}_{u1}^n + \left(\mathbf{F}_{p1}^n\right)^{-1}\mathrm{f}_{p1}^n\right\} \\ &+ \left(\mathbf{b}_6^n - \mathbf{b}_7^n\right)\frac{1}{2}\left\{\left(\mathbf{F}_{u0}^n\right)^{-1}\mathrm{f}_{u0}^n + \left(\mathbf{F}_{p0}^n\right)^{-1}\mathrm{f}_{p0}^n\right\} + \left(\mathbf{a}_5^n + \mathbf{a}_6^n\right)\frac{1}{2}\left\{\left(\tilde{\mathbf{F}}_{u1}^n\right)^{-1}\tilde{\mathrm{f}}_{u1}^n + \left(\tilde{\mathbf{F}}_{p0}^n\right)^{-1}\tilde{\mathrm{f}}_{p0}^n\right\} \\ &+ \left(\mathbf{a}_5^n - \mathbf{a}_6^n\right)\frac{1}{2}\left\{\left(\tilde{\mathbf{F}}_{u0}^n\right)^{-1}\tilde{\mathrm{f}}_{u0}^n + \left(\tilde{\mathbf{F}}_{p1}^n\right)^{-1}\tilde{\mathrm{f}}_{p1}^n\right\} + \mathbf{a}_7^n\left\{\left(\tilde{\mathbf{F}}_{u0}^n\right)^{-1}\tilde{\mathrm{f}}_{u0}^n + \left(\tilde{\mathbf{F}}_{p1}^n\right)^{-1}\tilde{\mathrm{f}}_{p1}^n\right\} - 2\left(\mathbf{a}_5^n + \mathbf{a}_7^n + \mathbf{b}_4^n + \mathbf{b}_6^n\right)\bar{\mathrm{f}}_n\end{aligned} \tag{3}$$

where the bar – or the tilde ~ on a quantity denotes the quantity averaged over the node volume or over an interface, respectively, **a** and **b** are constant matrices given by the cross-sections of node *n*, and **F** is a diagonal discontinuity matrix. All the heterogeneous interface and corner-point fluxes without hats on them are converted from their homogeneous partners using the relationships :

$\tilde{\mathrm{f}}_{\mathrm{di}}^{\mathrm{n}} = \tilde{\mathbf{F}}_{\mathrm{di}}^{\mathrm{n}}\hat{\tilde{\mathrm{f}}}_{\mathrm{di}}^{\mathrm{n}}$ and $\mathrm{f}_{\mathrm{di}}^{\mathrm{n}} = \mathbf{F}_{\mathrm{di}}^{\mathrm{n}}\hat{\mathrm{f}}_{\mathrm{di}}^{\mathrm{n}}$ for d=x,u,p and i=0,1.

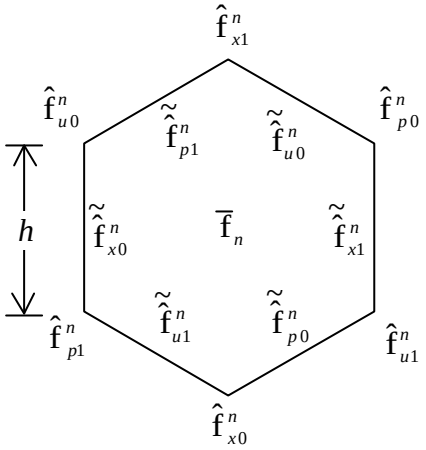

Figure 1. Hexagonal Node *n*

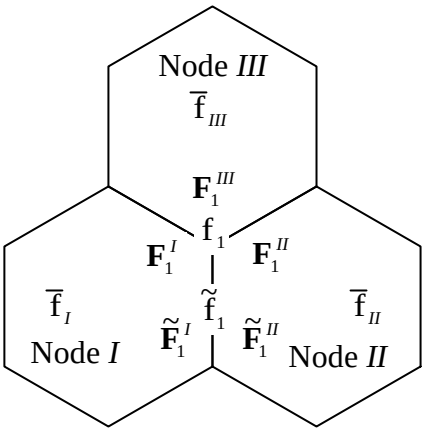

Figure 2. geometry for the derivation of nodal coupling equations

Then, the nodal neutron balance equation for node *I* shown in Figure 2 becomes :

$$\left\{ hS^n - \frac{4\sqrt{3}}{3}\left(\mathbf{a}_1^I + \mathbf{a}_3^I + \mathbf{b}_1^I + \mathbf{b}_3^I\right)\right\}\overline{f}_I = \frac{h}{k_{eff}}\chi S_f^I \overline{f}_I - \frac{2\sqrt{3}}{9}\left\{\left(\mathbf{a}_1^I + \mathbf{a}_3^I\right)\sum_{k=1}^{6}\left(\tilde{\mathbf{F}}_k^I\right)^{-1}\tilde{f}_\kappa + \left(\mathbf{b}_1^I + \mathbf{b}_3^I\right)\sum_{k=1}^{6}\left(\mathbf{F}_k^I\right)^{-1} f_\kappa\right\} \tag{4}$$

This equation is solved for the node-average flux.

The second coupling equation to be solved for the interface flux shared by nodes *I* and *II* can be obtained by applying the current continuity condition across the interface :

$$\tilde{\mathbf{J}}_{1I} = \tilde{\mathbf{J}}_{1II} \tag{5}$$

Finally, the coupling equation to be solved for the corner-point flux shared by nodes *I*, *II*, and *III* comes from the leakage balance around the corner point, which is given by

$$\mathbf{L}_{1I} + \mathbf{L}_{1II} + \mathbf{L}_{1III} = \mathbf{0} \tag{6}$$

The nodal balance equation (4) and two other coupling equations resulted from the conditions (5) and (6) are solved by an conventional inner/outer iteration scheme.

2.2 Mathematical Adjoint Equations

The mathematical adjoint flux is the solution of the equations obtained by transposing three different types of the forward coupling equations.

The coefficient matrices in the coupling equations may be interpreted as the response matrices between nodal unknowns. For example, the coefficient matrix on the corner-point flux f_1 in the nodal coupling equation for the node-average flux $\overline{f}_I$:

$$-\frac{2\sqrt{3}}{9}\left(\mathbf{b}_1^I + \mathbf{b}_3^I\right)\left(\mathbf{F}_1^I\right)^{-1} \tag{7}$$

may be interpreted as the response of the corner-point flux f_1 to the node-average flux $\overline{\mathrm{f}}_I$. A coefficient matrix in the forward equations and its transposed partner in the adjoint equations are completely opposite in the meaning of the response matrix. This means that the following matrix transposed from the matrix (7) will certainly become the coefficient matrix on the node-average adjoint flux $\overline{\mathrm{f}}_I^*$ in the coupling equation to be solved for the corner-point adjoint flux f_1^* in the adjoint system.

$$-\frac{2\sqrt{3}}{9}\left(\mathbf{F}_1^I\right)^{-1}\left(\mathbf{b}_1^I+\mathbf{b}_3^I\right)^T \tag{8}$$

So, in the adjoint system, this matrix appears at the position occupied by the matrix :

$$2\left(\mathbf{a}_5^I+\mathbf{a}_7^I+\mathbf{b}_4^I+\mathbf{b}_6^I\right)^T \tag{9}$$

in the forward corner-point leakage equation (3). By reversing the meaning of its counter part (7) in the forward flux euations, we can deduce the meaning of the transposed coefficient matrix (8) as the response of the node-average adjoint flux $\overline{\mathrm{f}}_I^*$ to the corner-point adjoint flux f_1^* in the adjoint flux equations. Collecting the terms having $\overline{\mathrm{f}}_I$ in them from all the coupling equations for the interface fluxes and the corner-point fluxes of node *I* and keeping the unknowns for which the coupling equations are solved, we can build the coupling equation for the node-average adjoint flux :

$$\begin{aligned}\left\{hS^n-\frac{4\sqrt{3}}{3}\left(\mathbf{a}_1^I+\mathbf{a}_3^I+\mathbf{b}_1^I+\mathbf{b}_3^I\right)\right\}^T\overline{\mathrm{f}}_I^* \\ =\frac{h}{k_{eff}}\left(\mathrm{cnS}_f^I\right)^T\overline{\mathrm{f}}_I^*+2\left\{\left(\mathbf{a}_1^I+\mathbf{a}_3^I+\mathbf{b}_1^I+\mathbf{b}_3^I\right)\sum_{k=1}^{6}\tilde{\mathrm{f}}_k^*+\left(\mathbf{a}_4^I+\mathbf{a}_6^I+\mathbf{b}_5^I+\mathbf{b}_7^I\right)\sum_{k=1}^{6}\mathrm{f}_\kappa\right\}\end{aligned} \tag{10}$$

This equation will never contain the discontinuity factors, because all the coefficient matrices of it come from the terms having $\overline{\mathrm{f}}_I$ in the forward coupling equations.

Likewise, collecting the terms having the interface flux $\tilde{\mathrm{f}}_1$ in all coupling equations for the forward nodal unknowns belonging to nodes *I* and *II* yields the coupling equation for the adjoint flux at the interface between nodes *I* and *II* shown in Figure 2:

$$\left(\tilde{\mathbf{F}}_1^I\right)^{-1}\tilde{\mathbf{J}}_{1I}^*=\left(\tilde{\mathbf{F}}_1^{II}\right)^{-1}\tilde{\mathbf{J}}_{1II}^* \tag{11}$$

where the adjoint current is defined on an interface of node *I* shown in Figure 1 as follows:

$$\begin{aligned}\tilde{\mathbf{J}}_{x0}^{*n}=&\left(\mathbf{a}_1^n+\mathbf{a}_2^n\right)^T\tilde{\mathrm{f}}_{x0}^{*n}+\left(\mathbf{a}_1^n-\mathbf{a}_2^n\right)^T\tilde{\mathrm{f}}_{x1}^{*n}+\left(\mathbf{a}_3^n+\mathbf{a}_4^n\right)^T\frac{1}{2}\left\{\tilde{\mathrm{f}}_{u1}^{*n}+\tilde{\mathrm{f}}_{p1}^{*n}\right\}+\left(\mathbf{a}_3^n-\mathbf{a}_4^n\right)^T\frac{1}{2}\left\{\tilde{\mathrm{f}}_{u0}^{*n}+\tilde{\mathrm{f}}_{p0}^{*n}\right\}\\&+\left(\mathbf{a}_5^n+\mathbf{a}_6^n\right)^T\frac{1}{2}\left\{\mathrm{f}_{u0}^{*n}+\mathrm{f}_{p1}^{*n}\right\}+\left(\mathbf{a}_5^n-\mathbf{a}_6^n\right)^T\frac{1}{2}\left\{\mathrm{f}_{u1}^{*n}+\mathrm{f}_{p0}^{*n}\right\}+\left(\mathbf{a}_7^n\right)^T\left\{\mathrm{f}_{u0}^{*n}+\mathrm{f}_{p1}^{*n}\right\}+\frac{2\sqrt{3}}{9}\left(\mathbf{a}_1^n+\mathbf{a}_3^n\right)^T\overline{\mathrm{f}}_n^*\end{aligned} \tag{12}$$

The equation (11) has a form of an adjoint interface current discontinuity. Note that this equation will contain the discontinuity factors of nodes *I* and *II* at the interface being evaluated, the inverses of which left-multiply all coefficient matrices. This seems natural, because all the coefficient matrices of this equation come from the terms containing $\tilde{\mathrm{f}}_1$ in the forward coupling equations.

Similarly, the coupling equation for the corner-point adjoint flux becomes

$$\left(\mathbf{F}_1^I\right)^{-1}\mathbf{L}_{1I}^*+\left(\mathbf{F}_1^{II}\right)^{-1}\mathbf{L}_{1II}^*+\left(\mathbf{F}_1^{III}\right)^{-1}\mathbf{L}_{1III}^*=\mathbf{0} \tag{13}$$

by defining the adjoint corner-point leakage as

$$\begin{aligned}\mathbf{L}_{x0}^{*n} &= \left(\mathbf{b}_4^n + \mathbf{b}_5^n\right)^T \mathrm{f}_{x0}^{*n} + \left(\mathbf{b}_4^n - \mathbf{b}_5^n\right)^T \mathrm{f}_{x1}^{*n} + \left(\mathbf{b}_6^n + \mathbf{b}_7^n\right)^T \frac{1}{2}\left\{\mathrm{f}_{u1}^{*n} + \mathrm{f}_{p1}^{*n}\right\} + \left(\mathbf{b}_6^n - \mathbf{b}_7^n\right)^T \frac{1}{2}\left\{\mathrm{f}_{u0}^{*n} + \mathrm{f}_{p0}^{*n}\right\} \\ &+ \left(\mathbf{b}_1^n + \mathbf{b}_2^n\right)^T \frac{1}{2}\left\{\tilde{\mathrm{f}}_{u1}^{*n} + \tilde{\mathrm{f}}_{p0}^{*n}\right\} + \left(\mathbf{b}_1^n - \mathbf{b}_2^n\right)^T \frac{1}{2}\left\{\tilde{\mathrm{f}}_{u0}^{*n} + \tilde{\mathrm{f}}_{p1}^{*n}\right\} + \left(\mathbf{b}_3^n\right)^T \left\{\tilde{\mathrm{f}}_{u0}^{*n} + \tilde{\mathrm{f}}_{p1}^{*n}\right\} + \frac{2\sqrt{3}}{9}\left(\mathbf{b}_1^n + \mathbf{b}_3^n\right)^T \bar{\mathrm{f}}_n^*\end{aligned} \tag{14}$$

Note that the equation (13) will have only the discontinuity factors of three nodes *I*, *II*, and *III* at the corner point.

The coupling equations for the mathematical adjoint fluxes are solved by the same iteration procedure for the forward fluxes, since they have the same structure as those for the forward fluxes. Therefore, their implementation is straightforward.

2.3 Physical Adjoint Equations

As mentioned in the introduction, the mathematical adjoint flux is required in perturbation theory. Therefore, there is no reason to try to get the physical adjoint flux if the mathematical adjoint flux is easily obtained. Here, however, we want to investigate how the physical adjoint equations are drived when the discontinuity factors are employed in nodal methods.

The adjoint diffusion equation which is the counter part of the forward diffusion equation (1) is given by

$$-\mathbf{D}^n \hat{\mathrm{f}}_n^*(x,y) + \left(\Sigma^n\right)^T \hat{\mathrm{f}}_n^*(x,y) = \frac{1}{k_{eff}} \left(\chi \nu \Sigma_f^n\right)^T \hat{\mathrm{f}}_n^*(x,y) \tag{15}$$

The physical adjoint flux is the solution of the coupling equations derived from this equation by the same method as the forward flux. Therefore, if there is no discontinuity factor involved, the physical adjoint is easily obtained by transposing the cross-sections.

When the discontinuity factors are adopted according to the equivalence theory, it is generally known that the physical adjoint flux is impossible or very difficult to obtain. Here, we successfully obtain the physical adjoint flux for the AFEN-type nodal methods. Assume that we build three types of the coupling equations for the mathematical adjoint fluxes from the adjoint diffusion equation. The coefficient matrices of these equations have the reverse meanings in a response matrix sense to those of their forward counter parts, since these equations are surely the equations for adjoint fluxes. For example, all the coefficient matrices in the coupling equation for the interface adjoint flux mean the responses of the unknowns comprising the node-average, the interface, and the corner-point adjoint fluxes to the interface adjoint flux being evaluated. Then, their counter parts in the forward coupling equations are the responses of the interface flux to the nodal unknowns involved. So they surely are all right-multiplied by the inverses of the interface discontinuity factors. Likewise the mathematical adjoint case, this, in turn, makes all the coefficient matrices in the interface adjoint flux equation eventually to be left-multiplied by the interface discontinuity factors of the nodes sharing the interface. The similar discussion made on the other two coupling equations indicates us that the node-average adjoint flux equation does not contain the discontinuity at all, and that the corner-point adjoint flux equation only contains the corner-point discontinuity factors of the three nodes sharing the corner point.

Recalling the forms of the mathematical ajoint equations in the previous section, this enables us to deduce that, when the equivalence theory is employed, the physical adjoint flux is equivalent to the solution of the adjoint diffusion equation :

$$-\mathbf{D}^n \mathrm{f}_n^*(x,y) + \left(\mathrm{S}^n\right)^T \mathrm{f}_n^*(x,y) = \frac{1}{k_{eff}} \left(\mathrm{cnS}_f^n\right)^T \mathrm{f}_n^*(x,y) \tag{16}$$

with the adjoint current discontinuity condition (not the flux discontinuity condition for the forward flux) and with the adjoint leakage imbalance condition :

$$\left(\tilde{\mathbf{F}}_1^I\right)^{-1} \hat{\tilde{\mathbf{J}}}_{1I}^* = \left(\tilde{\mathbf{F}}_1^{II}\right)^{-1} \hat{\tilde{\mathbf{J}}}_{1II}^* \quad , \tag{17}$$

$$\left(\mathbf{F}_1^I\right)^{-1} \hat{\mathbf{L}}_{1I}^* + \left(\mathbf{F}_1^{II}\right)^{-1} \hat{\mathbf{L}}_{1II}^* + \left(\mathbf{F}_1^{III}\right)^{-1} \hat{\mathbf{L}}_{1III}^* = \mathbf{0} \tag{18}$$

where we omit the hat ^ on the adjoint flux because it is not discontinuous any more, but, instead, we put it on the adjoint current and leakage. Physically, the current discontinuity at an interface and the leakage imbalance at a corner point mean that there are singular sources or absorbers at the interface and at the corner point, respectively.

The adjoint coupling equations derived from the adjoint diffusion equation (16) and the conditions of (17) and (18) will also have the same structure as their forward counter parts. This identical structure, again, enables us to use the same algorithm for the forward flux.

2.4 Two Ajoint Solutions of FDM in the One-Dimensional Rectangular Geometry

To investigate the properties of two adjoint fluxes in the case that the equivalence theory is employed, we compare them for FDM in the one-dimensional rectangular geometry.

One can easily derive the FDM expression for the forward current at the interface between node *n* and node *n*+1 shown in Figure 3, as follows.

$$\tilde{\mathbf{J}}_{n,n+1} = -\frac{2\mathbf{D}^n \mathbf{D}^{n+1}}{h\left(\mathbf{D}^n \tilde{\mathbf{F}}_0^{n+1} + \mathbf{D}^{n+1} \tilde{\mathbf{F}}_1^n\right)} \left(\tilde{\mathbf{F}}_0^{n+1} \overline{\mathrm{f}}_{n+1} - \tilde{\mathbf{F}}_1^n \overline{\mathrm{f}}_n\right) \tag{19}$$

Then, the forward equation for the node-average fluxes becomes

$$\begin{aligned} &-\frac{2\mathbf{D}^n \mathbf{D}^{n-1} \tilde{\mathbf{F}}_1^{n-1}}{h\left(\mathbf{D}^n \tilde{\mathbf{F}}_1^{n-1} + \mathbf{D}^{n-1} \tilde{\mathbf{F}}_0^n\right)} \overline{\mathrm{f}}_{n-1} - \frac{2\mathbf{D}^n \mathbf{D}^{n+1} \tilde{\mathbf{F}}_0^{n+1}}{h\left(\mathbf{D}^n \tilde{\mathbf{F}}_0^{n+1} + \mathbf{D}^{n+1} \tilde{\mathbf{F}}_1^n\right)} \overline{\mathrm{f}}_{n+1} \\ &+ \left\{ h\mathrm{S}^n + \frac{2\mathbf{D}^n \mathbf{D}^{n-1} \tilde{\mathbf{F}}_0^n}{h\left(\mathbf{D}^n \tilde{\mathbf{F}}_1^{n-1} + \mathbf{D}^{n-1} \tilde{\mathbf{F}}_0^n\right)} + \frac{2\mathbf{D}^n \mathbf{D}^{n+1} \tilde{\mathbf{F}}_1^n}{h\left(\mathbf{D}^n \tilde{\mathbf{F}}_0^{n+1} + \mathbf{D}^{n+1} \tilde{\mathbf{F}}_1^n\right)} \right\} \overline{\mathrm{f}}_n = \frac{h}{k_{eff}} \mathrm{cnS}_f^n \overline{\mathrm{f}}_n \end{aligned} \tag{20}$$

We write matrix operations on diffusion coefficients and discontinuity factors as scalar operations, because they are diagonal. Note that these equations are no longer symmetric, although they are symmetric when no discontinuity is involved.

With transposing Eq. (20), we can get the mathematical adjoint equation

$$-\frac{2\mathbf{D}^n\mathbf{D}^{n-1}\tilde{\mathbf{F}}_0^n}{h\left(\mathbf{D}^n\tilde{\mathbf{F}}_1^{n-1}+\mathbf{D}^{n-1}\tilde{\mathbf{F}}_0^n\right)}\bar{\mathrm{f}}_{n-1}^*-\frac{2\mathbf{D}^n\mathbf{D}^{n+1}\tilde{\mathbf{F}}_1^n}{h\left(\mathbf{D}^n\tilde{\mathbf{F}}_0^{n+1}+\mathbf{D}^{n+1}\tilde{\mathbf{F}}_1^n\right)}\bar{\mathrm{f}}_{n+1}^*$$
$$+\left\{h\left(\mathrm{S}^n\right)^T+\frac{2\mathbf{D}^n\mathbf{D}^{n-1}\tilde{\mathbf{F}}_0^n}{h\left(\mathbf{D}^n\tilde{\mathbf{F}}_1^{n-1}+\mathbf{D}^{n-1}\tilde{\mathbf{F}}_0^n\right)}+\frac{2\mathbf{D}^n\mathbf{D}^{n+1}\tilde{\mathbf{F}}_1^n}{h\left(\mathbf{D}^n\tilde{\mathbf{F}}_0^{n+1}+\mathbf{D}^{n+1}\tilde{\mathbf{F}}_1^n\right)}\right\}\bar{\mathrm{f}}_n^*=\frac{h}{k_{eff}}\left(\mathrm{cnS}_f^n\right)^T\bar{\mathrm{f}}_n^* \tag{21}$$

Deriving the physical adjoint equation starts from approximating the interface current as

$$\tilde{\hat{\mathbf{J}}}_{1n}^*=-\frac{2\mathbf{D}^n}{h}\left(\tilde{\mathrm{f}}_{n,n+1}^*-\bar{\mathrm{f}}_n^*\right) \tag{22}$$

Applying the interface adjoint current discontinuity condition given by

$$\left(\tilde{\mathbf{F}}_1^n\right)^{-1}\tilde{\hat{\mathbf{J}}}_{1n}^*=\left(\tilde{\mathbf{F}}_0^{n+1}\right)^{-1}\tilde{\hat{\mathbf{J}}}_{0n+1}^*\ , \tag{23}$$

solve it for the interface adjoint flux and current in terms of the node-average adjoint fluxes :

$$\tilde{\mathrm{f}}_{n,n+1}^*=\frac{\mathbf{D}^n\tilde{\mathbf{F}}_0^{n+1}\bar{\mathrm{f}}_n^*+\mathbf{D}^{n+1}\tilde{\mathbf{F}}_1^n\bar{\mathrm{f}}_{n+1}^*}{\mathbf{D}^n\tilde{\mathbf{F}}_0^{n+1}+\mathbf{D}^{n+1}\tilde{\mathbf{F}}_1^n}\ , \tag{24}$$

$$\tilde{\hat{\mathbf{J}}}_{1n}^*=-\frac{2\mathbf{D}^n\mathbf{D}^{n+1}\tilde{\mathbf{F}}_1^n}{h\left(\mathbf{D}^n\tilde{\mathbf{F}}_0^{n+1}+\mathbf{D}^{n+1}\tilde{\mathbf{F}}_1^n\right)}\left(\bar{\mathrm{f}}_{n+1}^*-\bar{\mathrm{f}}_n^*\right) \tag{25}$$

By integrating the adjoint diffusion equation (16) over the node volume and substituting Eq. (25) in it, we can get the physical adjoint equation which is completely identical to the mathematical adjoint equation (21). In spite of asymmetricity of the FDM forward equations having the discontinuity factors, the physical adjoint equation defined in the previous section becomes identical to the mathematical adjoint flux.

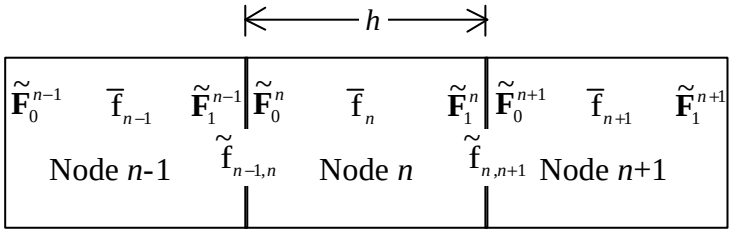

Figure 3. Array of one-dimensional rectangular nodes

3 Numerical Test Result

A benchmarking test for the physical and the mathematical adjoint flux models presented in this paper was performed for the core configuration I shown in Cho (1995A), whose results were reported elsewhere (Noh, 1999). This core consists of 151 heterogeneous hexagonal fuel assemblies with five different types of homogenized pincells whose pincell cross-sections are listed in the reference. The assembly homogenizations are performed by single-assembly fine mesh FDM calculations with zero current boundary conditions on the outer surfaces of the assembly. The homogenized assembly cross-sections are shown in Table I. One can see the surface and the corner-point discontinuity factors in the table.

Figure 4 compares the core multiplication factor and the node-average adjoint fluxes for two adjoint cases : the physical adjoint and the mathematical adjoint. Here, one can see only a 0.003 % error in the effective

multiplication factor and a 0.1% maximum error in the node-average adjoint fluxes. This good agreement tells the consistency in the definition of the physical adjoint flux.
We also found that the corner-point mathematical adjoint fluxes have alternating signs and very small absolute values, while the core-point physical adjoint fluxes are positive values with the magnitude of the node-average adjoint fluxes. (This phenomenon is also seen in the case that there is no discontinuity factor.) Therefore, in spite of its consistency in definition, it does not seem that the physical adjoint flux can be used as an approximation of the mathematical adjoint flux which is required in the perturbation theory.

Table 1. Homogenized Cross-sections of Each Assembly Type

Type	G	D (cm)	Σ_a (cm^{-1})	$\Sigma_{1\to2}$ (cm^{-1})	$\nu\Sigma_f$ (cm^{-1})	Interface D.F.[1]	Corner D.F.[1]
A	1	1.23099	0.0108167	0.0117134	0.0068032	0.995074	0.994444
	2	0.43165	0.0793297		0.1124590	1.113090	1.125876
B	1	1.23996	0.0114860	0.0109755	0.0083914	0.995695	0.995319
	2	0.42645	0.1000110		0.1546060	1.147650	1.161579
C	1	1.24486	0.0119430	0.0105623	0.0094394	0.996023	0.995788
	2	0.42177	0.1137300		0.1826250	1.171450	1.185583
D	1	1.24567	0.0004831	0.0322555	0.0	1.0	1.0
	2	0.25422	0.0157667		0.0	1.0	1.0

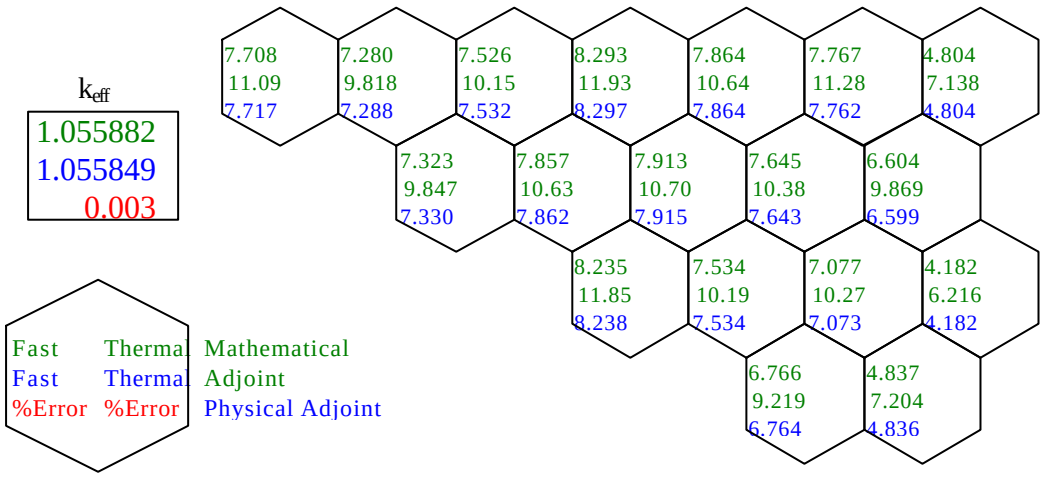

Figure 4 Results of Adjoint Flux Calculations for Core Configuration I

4 Conclusion

To use the perturbation formula in the AFEN-type nodal calculations, we develop the models to calculate the mathematical adjoint flux and the physical adjoint flux. By transposing three kinds of the forward nodal coupling equations, we derive the mathematical adjoint coupling equations with the coefficients whose meanings in a response matrix sense are quite opposite to those of the forward coupling equations. Although the mathematical adjoint flux is required in the perturbation theory, we also derive the physical

adjoint equations to disqualify the belief that it is impossible or nearly impossible to define the physical adjoint flux when the discontinuity factors are employed in a nodal method. The physical adjoint equations are derived from the adjoint diffusion equation with the adjoint current discontinuity conditions across the interfaces instead of the flux discontinuity conditions in the forward equations. We proved that two adjoint fluxes become identical for FDM, even when the discontinuity factors are involved.
The results of a numerical test show that the physical adjoint flux defined here is consistent to the mathematical adjoint flux even in the case that the interface and the corner-point discontinuity factors are different. However, it is also shown there are the considerable mismatches between two adjoint fluxes at corner points. Therefore, in spite of its consistency, it does not seem that the physical adjoint flux can be used as an approximation of the mathematical adjoint flux which is required in the perturbation theory.

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