Rayleigh-Taylor Instability of a Viscous Vapor-Liquid Interface with Mass Transfer

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1. Introduction

The Rayleigh-Taylor instability occurs when a heavier fluid overlies a lighter fluid in a gravitation field (Fig. 1). This instability plays an important role in the boiling heat transfer models for critical heat flux and film boiling. During the film boiling, the heat and mass transfer occur at the interface. However, numerous studies on the Rayleigh-Taylor instability have been done for isothermal flows. The studies considering the phase change are very limited.

[1] and [2] incorporated, for the first time, the phase change effect into the Rayleigh-Taylor instability for inviscid fluids. Those works showed that the evaporation had the stabilizing effect reducing the growth rate, whereas the condensation had the destabilizing effect. [3] presented the Rayleigh-Taylor instability of viscous fluids with heat and mass transfer. He showed that there was a strong stabilizing effect arising from the coupling viscosity and phase change. However, the analysis was made for fluids with equal kinematic viscosities. [4] also obtained the dispersion relation for viscous fluids with heat and mass transfer. The method was easier than [3], however, the fluids layers were semi-infinite in the extent, and they had equal kinematic viscosities.

For film boiling, the liquid layer can be considered to be semi-infinite, but the vapor layer is finite. The purpose of this study is to derive the dispersion relation for viscous fluids with heat and mass transfer. The vapor layer is finite while the liquid layer is semi-infinite. Two fluids have different properties. The critical and most unstable conditions are investigated considering the phase change effect.
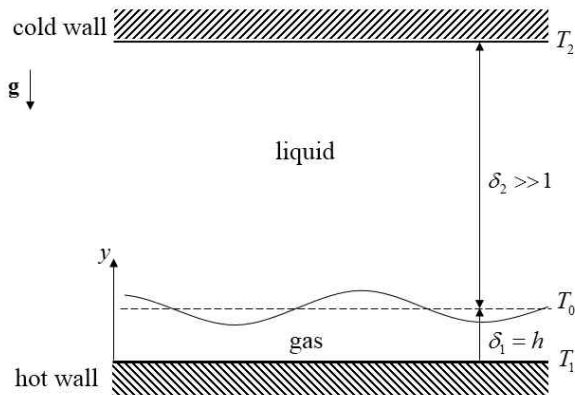

Fig. 1 Two-dimensional Rayleigh-Taylor instability of viscous gas-liquid interface with evaporation. The liquid layer is semi-infinite in the extent.

2. Rayleigh-Taylor Instability

A two-dimensional stability analysis is performed for incompressible viscous fluids with heat and mass transfer. The lighter fluid (vapor) is bounded by a heating wall and a heaver fluid (liquid). Two fluids are separated by a horizontal interface at is $y = \delta_1$. The base flow is motionless and thermally in an equilibrium state [1, 2]. All of the heat across the vapor layer from the heating wall go into the liquid layer without evaporation. The fluids are perturbed by a small disturbance with $\eta' = \tilde{\eta}e^{ikx+\omega t}$, where k is the wavenumber and ω is the growth rate. Substituting the perturbed quantities into the continuity and momentum equations, we obtain

$$\tilde{v}_1^{IV} - \left(2k^2 + \frac{\omega}{\nu_1}\right)\tilde{v}_1'' + k^2\left(k^2 + \frac{\omega}{\nu_1}\right)\tilde{v}_1 = 0 , \ (0 \le y \le h_0) \ (1)$$

$$\tilde{v}_2^{IV} - \left(2k^2 + \frac{\omega}{\nu_2}\right)\tilde{v}_2'' + k^2\left(k^2 + \frac{\omega}{\nu_2}\right)\tilde{v}_2 = 0 \ , \ (y \ge h_0) \ (2)$$

where the subscripts 1 and 2 indicate the vapor and liquid, respectively. $\tilde{v} = \tilde{v}(y)$ is the amplitude of the y velocity, and ν is the kinematic viscosity.

The no-slip velocity condition is applied to the wall:
$$\tilde{v}_1 = 0 , \ \tilde{v}_1' = 0 . \text{ (at } y = 0) \tag{3}$$
The liquid is unbounded. Thus,
$$\tilde{v}_2 \to 0 . \text{ (as } y \to \infty) \tag{4}$$
Below are the interfacial conditions.
$$\tilde{\eta} = \frac{\rho_1\tilde{v}_1}{b+\omega\rho_1} = \frac{\rho_2\tilde{v}_2}{b+\omega\rho_2} , \tag{5}$$
$$\tilde{v}_1' = \tilde{v}_2' , \tag{6}$$

$$\frac{\mu_1}{k^2}\tilde{v}_1''' - \left(3\mu_1 + \frac{\omega\rho_1}{k^2}\right)\tilde{v}_1' = \frac{\mu_2}{k^2}\tilde{v}_2''' - \left(3\mu_2 + \frac{\omega\rho_2}{k^2}\right)\tilde{v}_2' + (\sigma k^2 - \Delta\rho g)\frac{\rho_2\tilde{v}_2}{b+\omega\rho_2} . \tag{7}$$

$$\mu_1(\tilde{v}_1'' + k^2\tilde{v}_1) = \mu_2(\tilde{v}_2'' + k^2\tilde{v}_2) . \tag{8}$$

Equations (5) ~ (8) result from the mass/energy transfer matching, tangential velocity matching, normal stress balance, and shear stress matching, respectively. The variable b quantifies the degree of heat transfer at the base state as $G / (1/\delta_1 + 1/\delta_2)/L$, where L is the latent heat and G is $K_1(T_1 - T_0)/\delta_1 = K_2(T_0 - T_2)/\delta_2$. K is the thermal conductivity.

For a non-dimensional analysis, we define $m = \mu_2/\mu_1$, $r = \rho_2/\rho_1$, $\alpha = \delta_1 k$, $c = \omega\mu_1/(\Delta\rho g\delta_1)$, $F = \rho_1\Delta\rho g\delta_1^3/\mu_1^2$, $B = \Delta\rho g\delta_1^2/\sigma$, and $\beta = b\delta_1^2/\mu_1$.

The length and velocity are scaled by δ_1 and $\Delta \rho g \delta_1^2 / \mu_1$, respectively. The dimensionless form of Eqs (1) and (2) are given by

$$\bar{v}_1^{IV} - (\alpha^2 + p^2)\bar{v}_1'' + \alpha^2 p^2 \bar{v}_1 = 0, \quad (0 \le \bar{y} \le 1) \quad (9)$$

$$\bar{v}_2^{IV} - (\alpha^2 + q^2)\bar{v}_2'' + \alpha^2 q^2 \bar{v}_2 = 0, \quad (1 \le \bar{y}) \quad (10)$$

respectively. $\bar{v}$ and $\bar{y}$ are scaled variables, and $p^2 = \alpha^2 + Fc$ and $q^2 = \alpha^2 + rFc/m$. The wall and interfacial conditions through Eqs. (3)~(8) are, respectively, expressed as

$$\bar{v}_1 = 0, \ \bar{v}_1' = 0, \text{ (at } \bar{y} = 0) \quad (11)$$

$$\bar{v}_2 \to 0, \text{ (as } \bar{y} \to \infty) \quad (12)$$

$$\bar{v}_1 - \frac{r\beta + rFc}{\beta + rFc}\bar{v}_2 = 0, \quad (13)$$

$$\bar{v}_1' - v_2' = 0, \quad (14)$$

$$\bar{v}_1'' + \alpha^2 \bar{v}_1 - m(\bar{v}_2'' + \alpha^2 \bar{v}_2) = 0, \quad (15)$$

$$\bar{v}_1''' - (2\alpha^2 + p^2)\bar{v}_1' - m\bar{v}_2''' + (2m\alpha^2 + mq^2)\bar{v}_2' + \alpha^2(1 - B^{-1}\alpha^2)\frac{rF}{\beta + rFc}\bar{v}_2 = 0. \quad (16)$$

.

The general solution of (9) is

$$\bar{v}_1(\bar{y}) = A_{11}e^{-\alpha(\bar{y}-1)} + A_{12}e^{\alpha(\bar{y}-1)} + A_{13}e^{-p(\bar{y}-1)} + A_{14}e^{p(\bar{y}-1)}. \quad (17)$$

However, for the liquid to satisfy Eq. (12), the solution of Eq. (10) should be in the form

$$\bar{v}_2 = A_{21}e^{-\alpha(\bar{y}-1)} + A_{23}e^{-q(\bar{y}-1)}. \quad (18)$$

The form for the exponent is intended for easy applications of the interfacial conditions at $\bar{y} = 1$. Upon substitution of Eqs. (11)~(16) into Eqs. (17) and (18), we have a linear matrix system for A_{11}, A_{12}, A_{13}, A_{14}, A_{21},

$$A_2 B_2 = Rm(\alpha + q)[JT - mq^2 - m\alpha^2]B_2$$

$$+Rm\alpha(q - \alpha)\begin{Bmatrix} [(R+1)\alpha + m(q - \alpha)]2p\alpha \\ +[(R+1)\alpha + m(q - \alpha)]\alpha B_4 \\ +[2\alpha + m(q - \alpha)]pB_5 \\ -(\alpha + q)mB_2 \end{Bmatrix},$$

$$A_3 = R^2 m(\alpha^2 - p^2)p(\alpha + q),$$

$$A_4 = R^2(\alpha^2 + p^2)^2 + R(R+1)m\alpha p^2(q - \alpha),$$

$$A_5 = 2R^2\alpha^2 p[m(q - \alpha) + 2\alpha],$$

$$A_6 = \alpha p(\alpha^2 + p^2)[R(R+1)m(q - \alpha) + 4R^2\alpha],$$

$$B_1 = \alpha \cosh \alpha \sinh p - p \sinh \alpha \cosh p,$$

$$B_2 = 2\alpha p(1 - \cosh \alpha \cosh p) + (\alpha^2 + p^2)\sinh \alpha \sinh p,$$

$$B_3 = p \cosh \alpha \sinh p - \alpha \sinh \alpha \cosh p,$$

$$B_4 = \alpha \sinh \alpha \sinh p - p \cosh \alpha \cosh p,$$

$$B_5 = p \sinh \alpha \sinh p - \alpha \cosh \alpha \cosh p.$$

For isothermal flow ($\beta = 0$, $R = J = 1$), the dispersion relation reduces to the form of [5]. Physically, the wavenumber is real and positive, whereas the growth rate may be complex.

3. Results and Discussion

2.1 Effects of Evaporation

The dispersion relation, Eq. (21) was numerically

$$\begin{vmatrix} e^{\alpha} & e^{-\alpha} & e^{p} & e^{-p} & 0 & 0 \\ \alpha e^{\alpha} & -\alpha e^{-\alpha} & pe^{p} & -pe^{-p} & 0 & 0 \\ 1 & 1 & 1 & 1 & -R & -R \\ \alpha & -\alpha & p & -p & -\alpha & -q \\ 2\alpha^2 & 2\alpha^2 & \alpha^2 + p^2 & \alpha^2 + p^2 & -2m\alpha^2 & -m(\alpha^2 + q^2) \\ -(\alpha^2 + p^2) & (\alpha^2 + p^2) & -2p\alpha & 2p\alpha & m(\alpha^2 + q^2) - JT & 2\alpha qm - JT \end{vmatrix} = 0, \quad (19)$$

where $R = (r\beta + rFc)/(\beta + rFc)$, $J = rFc/(\beta + rFc)$, and $T = \alpha(1 - B^{-1}\alpha^2)/c$. The 6x6 matrix can reduce to the following 4x4 matrix using the determinant properties.

$$\begin{vmatrix} \sinh \alpha & \sinh p & R\cosh \alpha & R\cosh p \\ \alpha \cosh \alpha & p \cosh p & R\alpha \sinh \alpha & Rp \sinh p \\ m\alpha(\alpha + q) & pm(\alpha + q) & m\alpha^2 - qm\alpha - 2R\alpha^2 & -R(\alpha^2 + p^2) + m\alpha^2 - qm\alpha \\ -(\alpha^2 + p^2) + m\alpha^2 - qm\alpha & -2p\alpha + pm\alpha - pmq & mq(\alpha + q) - JT & mq(\alpha + q) - JT \end{vmatrix} = 0, \quad (20)$$

and A_{23}. For non-trivial solutions, the determinant of the matrix must be zero. Consequently,

Equation (20) leads to the dispersion relation:

$$V = A_1 B_1 + A B_2 + A_3 B_3 + A_4 B_4 + A_5 B_5 + A_6 = 0, \quad (21)$$

where

$$A_1 = R(\alpha^2 - p^2)[JT - mq(\alpha + q)],$$

solved using MATLAB. Figure 2 shows the growth rate c as a function of the wavenumber α for isothermal flows ($\beta = 0$). The fluid properties are those of saturated water and vapor at 150 bar. The vapor film thickness varies from $F = 1$ to $F = 10^8$. Figure 3 shows that when the evaporation decreases the growth rate. The effect of evaporation becomes more distinct for thin vapor films.
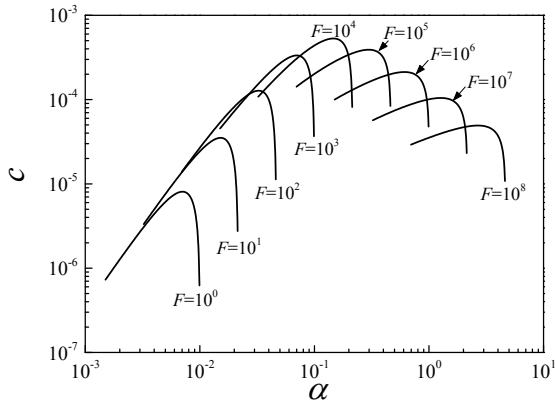
Fig. 2 Growth rate c for various thicknesses F of vapor layer. The phase change is not considered.
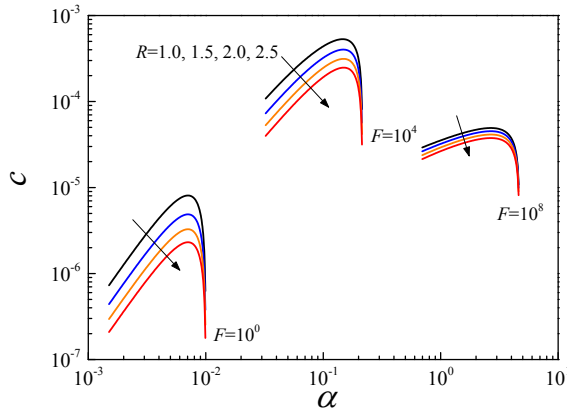

Fig. 3 Effect of phase change on the growth rate. The arrows indicate the direction of increasing evaporation.

2.2 Critical condition

The critical condition is the point at which the growth rate is zero. For wavenumbers lower than the critical wavenumber, the system is unstable, otherwise, stable. Since Eq. (21) becomes naturally zero when c is zero, the critical equation cannot be obtained directly by substituting $c = 0$ into Eq. (21). Instead, the critical equation is asymptotically obtained. If the growth rate c is infinitesimally small but not-zero, Eq. (21) can be expressed using the Taylor series expansion with regard to c.

$$V(c) = V(0) + V'(0)c + V''(0)c^2 / 2 + O(c^3) = 0 . \quad (22)$$

Since $V(0) = V'(0) = 0$, for Eq. (22) to be satisfied for non-zero c, $V''(0)$ must be zero. As result, we obtain the critical equation as

$$\beta = (1 - B^{-1}\alpha^2) \left[\frac{rF}{2\alpha} \frac{m(\cosh 2\alpha - 2\alpha^2 - 1) + (\sinh 2\alpha - 2\alpha)}{\left(\begin{array}{c} m^2(\cosh 2\alpha - 2\alpha^2 - 1) + m(\sinh 2\alpha - 2\alpha) \\ + mr\sinh 2\alpha + 2\alpha^2 r + 2\alpha mr + r\cosh 2\alpha + r \end{array} \right)} \right]$$
$$+ O(c). \quad (23)$$

The square bracket is always positive, since for $\alpha > 0$, $\cosh 2\alpha - 2\alpha^2 - 1 > 0$, $\sinh 2\alpha - 2\alpha > 0$, $m \geq 1$, and $r \geq 1$. If there is no phase change ($\beta = 0$), we reach $\alpha = \alpha_c = B^{1/2}$ (dimensionally, $k = k_c = (\Delta\rho g / \sigma)^{1/2}$). The RHS of Eq. (23) is positive only for $0 < \alpha < B^{1/2}$. If β is small, there are two values of α such that RHS equals β. The largest one is of physical importance. However, if β is higher than the maximum value of RHS, there is no solution of Eq. (23) with regard to α. In this case, the system is stable for wall wavenumbers.

For a thin layer of vapor, $\alpha = \delta_1 k$ is much smaller than unity. In this case, The RHS of (23) can be approximated using the Taylor series expansion with regard to α. As a result, we take

$$\beta = (1 - B^{-1}\alpha^2)\frac{1}{3}F\alpha^2 + O(c, \alpha^3) . \quad (24)$$

With dimensional variables, this equation is expressed as

$$k^4 - \frac{\Delta\rho g}{\sigma}k^2 + \frac{3v_1 b}{\sigma h_0^3} = 0 . \quad (25)$$

This is the critical equation for a thin vapor layer. When there is no phase change ($b = 0$), Eq. (25) yields $k_c = (\Delta\rho g / \sigma)^{1/2}$. If evaporation is so intensive that there is no solution for k in Eq. (25), the interface is stable for all wavenumbers. It is interesting to note that there is no liquid viscosity effect in Eq. (25). This can be explained by the fact that as the vapor layer is thin to the extent of creeping flow, the vapor viscosity becomes dominant.

On the other limit of a thick vapor layer ($\alpha \gg 1$), the hyperbolic functions are dominant over polynomials. Thus, Eq. (23) can be approximated as

$$\beta = (1 - B^{-1}\alpha^2) \left[\frac{rF}{2\alpha} \frac{m\cosh 2\alpha + \sinh 2\alpha}{m^2\cosh 2\alpha + m\sinh 2\alpha + mr\sinh 2\alpha + r\cosh 2\alpha} \right].$$
$$(26)$$

Arranging this,

$$2\alpha \left(m + r\frac{m\tanh 2\alpha + 1}{m + \tanh 2\alpha} \right)\beta = (1 - B^{-1}\alpha^2)rF . \quad (27)$$

Since $\tanh 2\alpha \to 1$ as $\alpha \to \infty$, the above equation can be written as

$$2\alpha(m + r)\beta = (1 - B^{-1}\alpha^2)rF . \quad (28)$$

In a dimensional form,

$$k^2 + \frac{2(v_1 + v_2)b}{\sigma}k - \frac{\Delta\rho g}{\sigma} = 0 . \quad (29)$$

Unlike Eq. (25), both liquid and vapor viscosities appear in the critical equation.

[3, 4] provided the critical equation for viscous fluids with equal kinematic viscosities and semi-infinite thicknesses.

$$k^2 + \frac{4\nu b}{\sigma}k - \frac{\Delta\rho g}{\sigma} = 0 . \tag{30}$$

One can see that Eq. (29) immediately reduces to (30) when $\nu = \nu_1 = \nu_2$. [3] claimed that the critical equation for a thin vapor layer was also given by Eq. (30). However, we found that there was error in deriving Eq. (30) for a thin vapor layer in [3]. If the error is corrected, the critical equation for a thin vapor layer becomes the same as Eq. (25). Consequently, the present critical equations are verified.

2.3 Most unstable condition

Figure 4 shows the variation of the most unstable wavelength λ_d for saturated water-vapor at 150 bar. For thin vapor layers (F is small), the evaporation does not affect λ_d, while for thick vapor layers, the evaporation increases λ_d.

To elucidate the behavior for thin vapor layers, the dispersion relation Eq. (31) is approximated using the Taylor series expansion to the second orders of F and β. Then, we obtain

$$
\begin{aligned}
c = &-\frac{1}{rF}\frac{m^2(\cosh 2\alpha - 2\alpha^2 - 1) + m(\sinh 2\alpha - 2\alpha) + mr\sinh 2\alpha + 2rm\alpha + 2\alpha^2 r + r\cosh 2\alpha + r}{m^2(\cosh 2\alpha - 2\alpha^2 - 1) + 2m\sinh 2\alpha + \cosh 2\alpha + 2\alpha^2 + 1}\beta \\
&+\frac{1 - B^{-1}\alpha^2}{2\alpha}\frac{m(\cosh 2\alpha - 2\alpha^2 - 1) + \sinh 2\alpha - 2\alpha}{m^2(\cosh 2\alpha - 2\alpha^2 - 1) + 2m\sinh 2\alpha + \cosh 2\alpha + 2\alpha^2 + 1} .
\end{aligned}
\tag{31}
$$

As seen, the evaporation reduces always the growth rate. For $\beta = 0$, the growth rate becomes zero at $\alpha = B^{1/2}$ (or $k = k_c = (\Delta\rho g / \sigma)^{1/2}$), and is maximized at $\alpha = (B/2)^{1/2}$ (or $k = k_d = (\Delta\rho g / (2\sigma))^{1/2}$. It is of interest to examine the asymptotic behavior for extremely thin layers of vapor. By the Taylor series expansion of Eq. (32) with regard to $\alpha = \delta_1 k \ll 1$,

$$c = -\frac{1}{F}\beta + \frac{1}{3}\alpha^2(1 - B^{-1}\alpha^2) + O(\alpha^3) . \tag{33}$$

The critical wavenumber is affected by β, whereas the most unstable wavenumber remains unchanged as $\alpha_d = (B/2)^{1/2}$.

Next, the most unstable condition for thick vapor layers without evaporation ($\beta = 0$) are examined. Figure 5 show the most unstable wavelength as a function of the saturation pressure for water-vapor. p_{cr} is the critical pressure. $\lambda_{c0} = 2\pi / (\sigma / (\Delta\rho g))^{1/2}$ is the critical wavelength for inviscid flows without phase change. For low pressures ($m \gg 1$ and $r \gg 1$), the most unstable

wavelength is very close to that by inviscid flows: $\lambda_d / \lambda_{c0} \approx 3^{1/2}$. However, the deviation between the viscous and inviscid flow analyses becomes apparent as the saturation pressure approaches the critical pressure. In general, the inviscid flow analysis have been performed for thick vapor layers. Figure 5 indicates that this approximation is no longer applicable near the critical pressure. However, it is reasonable to use the inviscid flow approximation in the wide range of pressure.
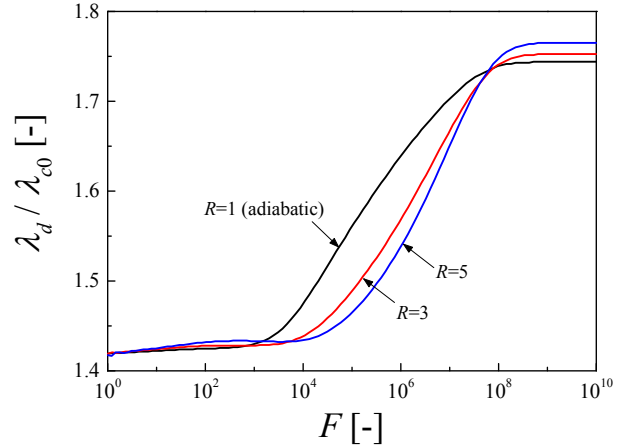

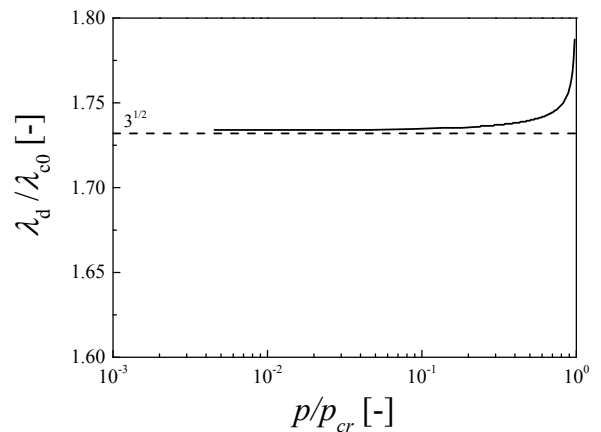**Fig. 4** Effects of the evaporation on the most unstable wavelength

Fig. 5 Variation of λ_d with the saturation pressure of water. The fluid layers are semi-infinite.

3. Conclusions

A linear stability analysis has been performed for the Rayleigh-Taylor instability of viscous fluids with phase

change. A semi-infinite layer of liquid superposed on a finite layer of vapor is considered. At the base flow, the direction of heat transfer is from vapor toward liquid. In this case, the phase change always reduces the growth rate. For a thin layer of vapor, the critical equation is

$$k^4 - \frac{\Delta \rho g}{\sigma} k^2 + \frac{3\nu_1 b}{\sigma \delta_1^3} = 0 \, .$$

There is a coupled effect of the vapor viscosity, phase change, and vapor layer thickness on the critical wavenumber. For a very thick layer of vapor, the critical equation is

$$k^2 + \frac{2(\nu_1 + \nu_2)b}{\sigma} k - \frac{\Delta \rho g}{\sigma} = 0 \, .$$

The effect of the vapor layer thickness is weakened and both liquid and vapor viscosities affect the critical wavenumber. As to the most dangerous wavelength, for a thin layer of vapor, it is given by $\lambda_d / \lambda_{c0} = 2^{1/2}$ (or $\lambda_d = 2\pi(2\sigma / (\Delta \rho g))^{1/2}$), and is insensitive to the degree of phase change. However, for a thick layer of vapor, the most dangerous wavelength increases with the degree of phase change.

Acknowledgment

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