Fourier Stability Analysis of 1-D/1-D Transport-Transport Calculation and 1-D/1-D Transport-Diffusion Approximation

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1. Introduction

In recent years, 3-D whole-core transport calculation is drawing an increasing interest in reactor analysis due to the needs of accurate solution and the increase of computing power. But, a direct 3-D transport calculation without spatial homogenization still requires a huge computational burden. To deal with 3-D heterogeneous reactor problems avoiding a direct 3-D calculation, a 2-D/1-D fusion method [1–3] was developed at KAIST. This method employs 2-D MOC calculation in the radial direction and 1-D S_N calculation in the axial direction. The 2-D/1-D fusion method gives accurate transport solutions for several 3-D reactor problems. On the other hand, a hybrid method which employs 2-D MOC calculation in the radial direction and 1-D diffusion (or diffusion-like) approximation in the axial direction was developed by another research group [4, 5]. The above method gives more accurate solution than diffusion calculation, but there is some difference from transport solution due to the 1-D diffusion approximation.

Although the methods were developed a few years ago, the question of the stability of the methods was not answered satisfactorily. In [6, 7], it is reported that the latter method [4, 5] exhibits unstable behavior for small axial mesh sizes and that this behavior can be resolved by under-relaxation. But additional studies are necessary because there is no axial calculation scheme in [6, 7]. Moreover, the convergent result is not the same as the transport solution [7]. For the 2-D/1-D fusion method, only a refinement sensitivity study via numerical calculation is available in [8].

This paper presents the stability of the above methods via Fourier stability analysis (FSA). The conditions of convergence are also verified.

2. Theory and Methods

In this section, the two methods; transport-transport calculation and transport-diffusion approximation are briefly described.

2.1 Basic Assumptions

A goal of this paper is the verification of relations between the stability and the axial mesh size. To concentrate and achieve the goal, the following simplifications which do not harm the convergence properties are introduced.

First, a 2-D transport equation is considered by reducing one dimension in the radial direction (now, “radial” is changed into “horizontal”, “axial” is changed into “vertical”). Second, horizontal fine meshes are very small so that the spatially continuous equation is considered in the horizontal direction.

2.2 Transport-Transport Calculation

Let us first consider a one-group neutron transport equation on 2-D infinite homogeneous medium with isotropic scattering:

$$\left[\Omega_x \frac{\partial}{\partial x} + \Omega_z \frac{\partial}{\partial z} + \sigma_t\right]\psi^{(l+1)}(\vec{\Omega},\vec{r}) = \sigma_s \phi^{(l)}(\vec{r}) + S. \quad (1)$$

The equations of a 2-D version of 2-D/1-D fusion method (denoted “transport-transport”, “TT”) are obtained by splitting of directional operators and directional integration:

$$\begin{aligned}\left[\Omega_x \frac{\partial}{\partial x} + \sigma_t\right]\psi_k^{(l+1/2)} &= \sigma_s \phi_k^{(l)} + S \\ &\quad - \frac{\Omega_z}{h_z}\left(\psi_{k+1/2}^{(l)} - \psi_{k-1/2}^{(l)}\right),\end{aligned} \quad (2)$$

$$\begin{aligned}\frac{\Omega_z}{h_z}\left(\psi_{k+1/2}^{(l+1)} - \psi_{k-1/2}^{(l+1)}\right) + \sigma_t \psi_k^{(l+1)} &= \sigma_s \phi_k^{(l+1/2)} + S \\ &\quad - \Omega_x \frac{\partial \psi_k^{(l+1/2)}}{\partial x},\end{aligned} \quad (3)$$

where $\psi_k^{(l+1/2)} = \int_{z_{k-1/2}}^{z_{k+1/2}} \psi^{(l+1/2)}(\vec{\Omega},\vec{r})dz$ and k is the vertical mesh index. The above equations are iterative between the horizontal and the vertical calculations. Because the equations (2) and (3) are solved by transport methods, the result is a 2-D solution based on the transport equation. An auxiliary discretization scheme is necessary for the vertical equation (3). The present paper uses three schemes; diamond difference (DD), step characteristic (SC), and linear characteristic (LC).

2.3 Transport-Diffusion Approximation

In [6], the vertical transport leakage term in Eq. (1) is approximated by diffusion leakage and it moves to the right hand side. The resulting equation is

$$\left[\Omega_x \frac{\partial}{\partial x} + \sigma_t\right]\psi^{(l+1)}(\vec{\Omega},\vec{r}) = \sigma_s\phi^{(l)}(\vec{r}) + S + \frac{1}{3\sigma_t}\frac{\partial^2\phi^{(l)}(\vec{r})}{\partial z^2}. \quad (4)$$

The vertical integration of Eq. (4) becomes

$$\left[\Omega_x \frac{\partial}{\partial x} + \sigma_t\right]\psi_k^{(l+1/2)} = \sigma_s\phi_k^{(l)} + S + \frac{1}{3\sigma_t h_z^2}\left(\phi_{k+1}^{(l)} - 2\phi_k^{(l)} + \phi_{k-1}^{(l)}\right), \quad (5)$$

which is the horizontal equation to be solved.

There is no vertical calculation in [6] (will be denoted "Half TD"). The present paper uses an additional equation for vertical calculation (denoted "Basic TD"):

$$-\frac{1}{3\sigma_t h_z^2}\left(\phi_{k+1}^{(l+1)} - 2\phi_k^{(l+1)} + \phi_{k-1}^{(l+1)}\right) + \sigma_a\phi_k^{(l+1)} = S - \int \Omega_x \frac{\partial\psi_k^{(l+1/2)}}{\partial x} d\vec{\Omega}. \quad (6)$$

The above equations (5) and (6) are iterative between the horizontal and the vertical calculations. Because the diffusion leakage source is used in the equation (5) and the equation (6) is solved by the diffusion method, the result is not a 2-D transport solution, although it should be more accurate than 2-D diffusion approximation.

3. Stability Analysis

In this section, the applications of FSA to the two methods are described. Let us set the following Fourier ansatz [6]:

$$\begin{aligned} &S \to 0, \\ &\psi_k^{(l)} \to \omega^l b(\vec{\Omega}) e^{i\sigma_t(\vec{\lambda}\cdot\vec{r})}, \\ &\psi_k^{(l+1/2)} \to \omega^l a(\vec{\Omega}) e^{i\sigma_t(\vec{\lambda}\cdot\vec{r})}. \end{aligned} \quad (7)$$

Substituting Eq. (7) into the governing equations, the spectral radius can be obtained. The stability and the convergence rate are determined by the spectral radius.

3.1 Transport-Transport Calculation

The spectral radii of the transport-transport calculation are obtained as follows:

$$\rho_{\mathrm{TT(DD)}} = 1, \quad (8)$$

$$\rho_{\mathrm{TT(SC)}} = \max\left\{c, \max_{\Omega_z}\left(\frac{\tanh(\frac{\sigma_t h_z}{2\Omega_z})}{\frac{\sigma_t h_z}{2\Omega_z}}\right)\right\} < 1, \quad (9)$$

where c is the scattering ratio. The spectral radius of TT with the DD scheme is equal to 1. For the SC scheme, the iteration is stable and converging because the spectral radius is less than 1. For the LC scheme, there is no explicit solution, but implicitly given:

$$\rho_{\mathrm{TT(LC)}} = \max\left\{c, \tilde{\omega}\right\}, \quad (10)$$

where eigenvalue $\tilde{\omega}$ satisfies the following eigenvalue equation:

$$\begin{aligned} &\left[\frac{1+\exp(-\sigma_t h_z/\Omega_z)}{1-\exp(-\sigma_t h_z/\Omega_z)}\right]\tilde{\omega} b(\vec{\Omega}) = \frac{2\Omega_z}{\sigma_t h_z} b(\vec{\Omega}) \\ &\quad + c\left[\frac{1+\exp(-\sigma_t h_z/\Omega_z)}{1-\exp(-\sigma_t h_z/\Omega_z)} - \frac{2\Omega_z}{\sigma_t h_z}\right]\int b(\vec{\Omega})d\vec{\Omega} \\ &\quad + c\int\left[\frac{1+\exp(-\sigma_t h_z/\Omega_z)}{1-\exp(-\sigma_t h_z/\Omega_z)} - \frac{2\Omega_z}{\sigma_t h_z}\right] b(\vec{\Omega})d\vec{\Omega}. \end{aligned} \quad (11)$$

Fig. 1 is a comparison of FSA theoretic results (th) and numerical results (num) of the transport-transport calculation. Numerical calculation is performed under the following conditions; S_8, fixed number of cells (200×200), fixed horizontal-direction cell size (0.1cm), and periodic boundary condition.

Numerical results are close to the FSA theoretic results except for the LC scheme. The difference between FSA theoretic result and numerical result for the LC scheme is not significant. For the DD scheme, the convergence speed is very slow, almost neutrally stable. For the SC and the LC schemes, the spectral radii are close to 1 for small vertical mesh sizes. But the transport-transport calculation (and 2-D/1-D fusion method implied) with the SC and LC schemes is unconditionally stable because the spectral radius is less than 1.

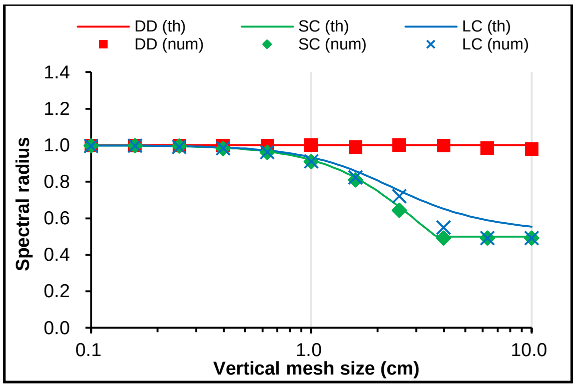

Fig. 1. Spectral radii of the transport-transport calculation (c = 0.5).

3.2 Transport-Diffusion Approximation

The spectral radii of the transport-diffusion approximation are obtained as follows:

$$\rho_{\mathrm{TD(Half)}} = \max\left\{\frac{1}{3(\frac{\sigma_t h_z}{2})^2} - c, c\right\}, \tag{12}$$

$$\rho_{\mathrm{TD(Basic)}} = \max\left\{\frac{c}{1-c}, \frac{1-3c(\frac{\sigma_t h_z}{2})^2}{1+3(1-c)(\frac{\sigma_t h_z}{2})^2}\right\}. \tag{13}$$

Basic TD diverges if the scattering ratio is larger than 0.5. It is a serious problem (worse than Half TD in [6]). To resolve the above problem, the present paper proposes the scattering source update in the vertical calculation (denoted "Scattering TD"):

$$-\frac{1}{3\sigma_t h_z^2}\left(\phi_{k+1}^{(l+1)} - 2\phi_k^{(l+1)} + \phi_{k-1}^{(l+1)}\right) + \sigma_t \phi_k^{(l+1)} = \sigma_s \phi_k^{(l+1/2)} + S - \int \Omega_x \frac{\partial \psi_k^{(l+1/2)}}{\partial x} d\vec{\Omega}, \tag{14}$$

that is a modification of Eq. (6). The spectral radius is then changed to

$$\rho_{\mathrm{TD(Scattering)}} = \max\left\{c, \frac{1-3c(\frac{\sigma_t h_z}{2})^2}{1+3(\frac{\sigma_t h_z}{2})^2}\right\}, \tag{15}$$

which says that Scattering TD is unconditionally stable.

Fig. 2 is a comparison of FSA theoretic results (th) and numerical results (num) of the transport-diffusion approximation. The conditions of numerical calculation are the same as those of TT calculation.

Numerical results are close to the theoretic results of FSA. In Fig. 2, Basic TD shows neutral stability. But if the scattering ratio is larger than 0.5, Basic TD becomes unstable. Half TD is also unstable for small vertical mesh sizes. For the Scattering TD, the spectral radius increases as the size of vertical mesh decreases. But the Scattering TD is unconditionally stable because the spectral radius is still less than 1.

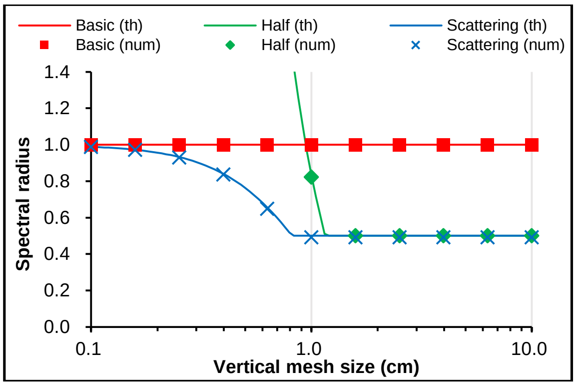

Fig. 2. Spectral radii of the transport-diffusion approximation (c = 0.5)

4. Conclusions

In this paper, the Fourier stability analysis of the transport-transport calculation (2-D/1-D fusion method) and the transport-diffusion approximation (2-D transport and 1-D diffusion approximation) is presented. The transport-transport calculation is unconditionally stable for vertical (axial) SC and LC schemes. The transport-diffusion approximation exhibits unstable behaviors. To resolve this instability problem, scattering source update in the vertical calculation is proposed in this paper.

The related topics of acceleration and modular cell homogenization are not considered in this analysis, that are left as future studies.

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