Convergence Analysis of Two-Node CMFD Method for Two-Group Neutron Diffusion Eigenvalue Problem

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1. Introduction

The nonlinear coarse-mesh finite difference method with two-node local problem (CMFD2N) [1] is proven to be unconditionally stable for neutron diffusion eigenvalue problem. The explicit expression of current correction factor (CCF) has been derived based on the analytic two-node nodal method (ANM2N) and Fourier analysis is applied to the linearized algorithm [2]. There have been several successes applying the Cefus and Larsen's approach to algorithms with fixed source neutron diffusion problems [3-8]. It is shown that the analytic convergence rate of CMFD2N for neutron diffusion eigenvalue problem obtained by Fourier analysis compares very well with the numerically measured convergence rate [9].

2. Computational methodology

The model problem is a 1-D 2-G neutron diffusion eigenvalue problem in an infinite homogeneous medium. The neutron diffusion equation for the model problem is written as:

$$\begin{aligned} -D_1\frac{d^2}{dx^2}\phi_1(x)+\Sigma_{r1}\phi_1(x)&=\frac{1}{k_{eff}}(\nu\Sigma_{f1}\phi_1(x)+\nu\Sigma_{f2}\phi_2(x)),\\ -D_2\frac{d^2}{dx^2}\phi_2(x)+\Sigma_{a2}\phi_2(x)&=\Sigma_{s12}\phi_1(x). \end{aligned} \quad (1)$$

The cross section data set used for the model problem is taken from the composition #4 base cross section of the NEACRP LWR transient benchmark [10]. The boundary conditions are periodic for both boundaries.

The derivation of the two-node analytic nodal kernel can be easily found in the literature [11]. In the CMFD2N algorithm for 1-D 2-G model problem, the CCF for each node interface, $\hat{D}_{g,i}$, can be defined by the following equation

$$J_{g,i}^{ANM\,2N}=-(D_g/h)(\bar{\phi}_{g,i+1}-\bar{\phi}_{g,i})+\hat{D}_{g,i}(\bar{\phi}_{g,i+1}+\bar{\phi}_{g,i}). \quad (2)$$

It is possible to directly obtain CCFs for each local problem from CMFD2N variables by manipulating net current solution of ANM2N, as following equation

$$\begin{aligned} \hat{D}_{1,i}^{(n)}&=\frac{(C_1^{(n-1)}+D_1/h)(\bar{\phi}_{1,i+1}^{(n-1)}-\bar{\phi}_{1,i}^{(n-1)})+C_2^{(n-1)}(\bar{\phi}_{2,i+1}^{(n-1)}-\bar{\phi}_{2,i}^{(n-1)})}{\bar{\phi}_{1,i+1}^{(n-1)}+\bar{\phi}_{1,i}^{(n-1)}},\\ \hat{D}_{2,i}^{(n)}&=\frac{C_3^{(n-1)}(\bar{\phi}_{1,i+1}^{(n-1)}-\bar{\phi}_{1,i}^{(n-1)})+(C_4^{(n-1)}+D_2/h)(\bar{\phi}_{2,i+1}^{(n-1)}-\bar{\phi}_{2,i}^{(n-1)})}{\bar{\phi}_{1,i+1}^{(n-1)}+\bar{\phi}_{1,i}^{(n-1)}}, \end{aligned} \quad (3)$$

where n is an iteration index, and $C_p^{(n-1)}$ $(p=1,2,3,4)$ is a coefficient dependent on buckling in $(n-1)th$ iteration.

The global coupled coefficient matrix for CMFD2N can be derived by integrating Eq. (1) and applying CCFs of Eq. (2).

3. Convergence analysis

In this section, numerically measured convergence rates and analytically derived convergence rates will be compared each other.

3.1. Numerical convergence analysis

It is empirically known that the convergence rate of the CMFD2N algorithm is governed by the convergence rate of CCFs, i.e., once the CCFs converge, then the solution will converge in one iteration [6]. The numerical convergence rate can be measured by the following formula.

$$\rho=\frac{\|\hat{D}^{(n)}-\hat{D}^{(n-1)}\|_2}{\|\hat{D}^{(n-1)}-\hat{D}^{(n-2)}\|_2}. \quad (4)$$

Fig. 1 presents numerically measured convergence rates. It should be noted that, when the number of nodes is even, the convergence rate is independent of the number of nodes, while it depends on the mesh size. In case of odd number of nodes, the convergence rate depends not only on the mesh size but also slightly on the number of nodes. Interestingly, however, it converges to that in the even number of nodes case as

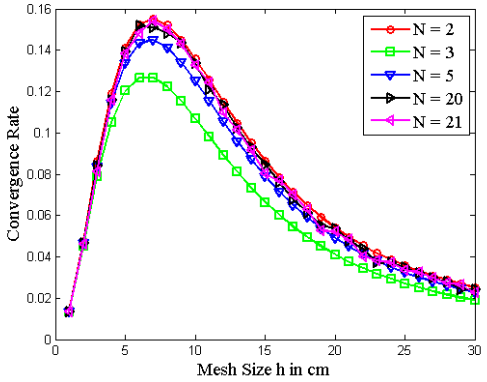

Fig. 1. Numerical Convergence Rate

3.2. Analytic derivation of convergence rate

It is interesting to note from the numerical measurements that the convergence rate of CMFD2N is closely related with mesh size. Therefore, in this section, we are interested in the relationship between the convergence rate and the mesh size. In the numerical measurements, the outer and inner iterations of the power method with CMFD matrix are fully converged in order to obtain a consistent convergence rate with analytical approach, which means that the numerically obtained eigenvalue and eigenvector set actually correspond to the exact ones of CMFD matrices with the corresponding CCFs. This property makes it possible to generalize the convergence analysis from a small to a large number of nodes because the algorithm is only dependent on the mesh size but not on the total problem size. For a small matrix, it is possible to derive eigenvalue and eigenvector using symbolic manipulations. Eq. (5) is a generalized form of eigenvalue and eigenvector for the problem whose total number of nodes is only two.

$$\begin{aligned}
&\bar{\phi}_{1,i}^{(n)} = \phi_{1,0}(1+\frac{h}{2}\frac{\Sigma_{a2}k_\infty(4D_2+h^2\Sigma_{a2})(\hat{D}_{1,i-1}^{(n)}-\hat{D}_{1,i}^{(n)})+h^2\nu\Sigma_{f2}\Sigma_{s12}(\hat{D}_{2,i-1}^{(n)}-\hat{D}_{2,i}^{(n)})}{\Sigma_{a2}k_\infty(4D_2+h^2\Sigma_{a2})D_1+h^2\nu\Sigma_{f2}\Sigma_{s12}D_2}),\\
&\bar{\phi}_{2,i}^{(n)} = \phi_{2,0}(1+\frac{h}{2}\frac{h^2\Sigma_{a2}^2k_\infty(\hat{D}_{1,i-1}^{(n)}-\hat{D}_{1,i}^{(n)})+(4\Sigma_{a2}k_\infty D_1+h^2\nu\Sigma_{f2}\Sigma_{s12})(\hat{D}_{2,i-1}^{(n)}-\hat{D}_{2,i}^{(n)})}{h^2\Sigma_{a2}^2k_\infty D_1+(4\Sigma_{a2}k_\infty D_1+h^2\nu\Sigma_{f2}\Sigma_{s12})D_2}),\\
&k_{eff}^{(n)} = k_\infty.
\end{aligned} \quad (5)$$

First-order perturbations of $\bar{\phi}_{g,i}^{(n)}$, $\hat{D}_{g,i}^{(n)}$, and $C_p^{(n)}$ ($g=1,2$, $p=1,2,3,4$) are introduced because they depend on the iteration steps:

$$\bar{\phi}_{g,i}^{(n)} = \phi_{g,0}(1+\varepsilon\xi_{g,i}^{(n)}),\ \hat{D}_{g,i}^{(n)} = \varepsilon\tau_{g,i}^{(n)}, \text{ and } C_p^{(n)} = C_{p,0}(1+\varepsilon\theta_p^{(n)}). \quad (6)$$

Substituting Eq. (6) into Eqs. (3) and (5), and dropping the $O(\varepsilon^2)$ terms yields the linearized equations.

If the dimension of the flux vector is *N*, there are *N* independent bases for the flux vector. The flux vector can be expanded by the *N* eigenvectors $e^{i\lambda_m x}$ $(m=0,1,...,N-1)$, corresponding to the eigenmodes $\lambda_m = 2m\pi/(Nh)$. By using the *N* base vectors, the following Fourier ansatz can be introduced [8].

$$\begin{aligned}
\xi_{g,i}^{(n)} &= \xi_{g,i,m}^{(n)} = a_{g,m}\omega_m^n e^{i\lambda_m x_{i-1/2}},\\
\tau_{g,i}^{(n)} &= \tau_{g,i,m}^{(n)} = b_{g,m}\omega_m^n e^{i\lambda_m x_i}.
\end{aligned} \quad (7)$$

For $m=0$, the linearized algorithm yields $\omega_0 = 0$. For $1 \le m \le N-1$, the linearized algorithm yields

$$\begin{aligned}
&\omega_m b_{1,m} = (C_{1,0}+\frac{D_1}{h})\sinh(i\lambda_m h/2)a_{1,m} + C_{2,0}\frac{\Sigma_{s12}}{\Sigma_{a2}}\sinh(i\lambda_m h/2)a_{2,m},\\
&\omega_m b_{2,m} = C_{3,0}\frac{\Sigma_{a2}}{\Sigma_{s12}}\sinh(i\lambda_m h/2)a_{1,m} + (C_{4,0}+\frac{D_2}{h})\sinh(i\lambda_m h/2)a_{2,m},\\
&a_{1,m} = -h\frac{\Sigma_{a2}k_\infty(4D_2+h^2\Sigma_{a2})\sinh(i\lambda_m h/2)b_{1,m} + h^2\nu\Sigma_{f2}\Sigma_{s12}\sinh(i\lambda_m h/2)b_{2,m}}{\Sigma_{a2}k_\infty(4D_2+h^2\Sigma_{a2})D_1+h^2\nu\Sigma_{f2}\Sigma_{s12}D_2},\\
&a_{2,m} = -h\frac{h^2\Sigma_{a2}^2k_\infty\sinh(i\lambda_m h/2)b_{1,m} + (4\Sigma_{a2}k_\infty D_1+h^2\nu\Sigma_{f2}\Sigma_{s12})\sinh(i\lambda_m h/2)b_{2m}}{h^2\Sigma_{a2}^2k_\infty D_1+(4\Sigma_{a2}k_\infty D_1+h^2\nu\Sigma_{f2}\Sigma_{s12})D_2},
\end{aligned} \quad (8)$$

and after some manipulations, we obtain

$$\omega_m = \frac{1-(\mu_0 h/2)^2\,\mathrm{csch}^2(\mu_0 h/2)}{1+(\mu_0 h/2)^2}\sin^2(m\pi/N), \quad (9)$$

where μ_0 is a converged second harmonic buckling:

$$\mu_0 = \sqrt{\frac{\Sigma_{a2}}{D_2}+\frac{\nu\Sigma_{f2}\Sigma_{r1}\Sigma_{s12}}{D_2\nu\Sigma_{f1}\Sigma_{a2}+D_1\nu\Sigma_{f2}\Sigma_{s12}}}. \quad (10)$$

The analytic convergence rate of the linearized algorithm for this problem is governed by maximum ω_m. The analytic convergence rate for *N=2* can be simplified as

$$\rho_{N=2} = \max_{m=0,1,...,N-1}|\omega_m| = \frac{1-(\mu_0 h/2)^2\,\mathrm{csch}^2(\mu_0 h/2)}{1+(\mu_0 h/2)^2}. \quad (11)$$

Without loss of generality, it is possible to write an analytic convergence rate for three nodes as Eq. (12).

$$\rho_{N=3} = \frac{1-(\mu_0 h/2)^2\,\mathrm{csch}^2(\mu_0 h/2)}{3/4+(\mu_0 h/2)^2}\sin^2(\pi/3). \quad (12)$$

4. Results

In Fig. 2, the lines and the dots represent analytic and numerical convergence rates, respectively.

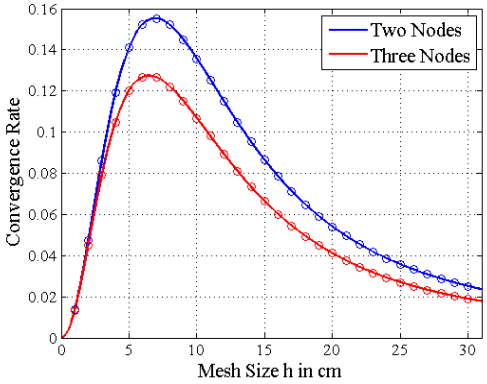

Fig. 2. Numerical and Analytic Convergence Rates for Two and Three Nodes

It should be noted that the numerically measured convergence rate and the analytically derived

convergence rate match each other exactly for the two-node and three-node problems, and also that the convergence is unconditionally stable for all mesh sizes.

Following analysis can be directly applied to three nodes problem. In the two nodes problem and the given cross section data, the $h_{\max}$ can be obtained as 6.904, and the corresponding convergence rate is 0.155. In Fig. 3, the convergence rate of the CMFD2N algorithm is plotted as a function of (μ_0, h). As shown in the figure, the convergence rate increases first and decreases after the maximum point as h and/or μ_0 increases. The maximum convergence rate is always 0.155 and satisfies $\mu_0 h = 2.864$ regardless of cross section data.

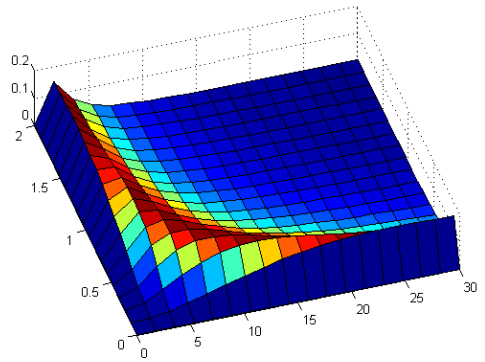

Fig. 3. Convergence rate of two nodes versus (μ_0, h)

5. Conclusions

The convergence behavior of the nonlinear coarse-mesh finite difference method with two-node local problem solved by ANM2N kernel for the homogeneous one-dimensional two-group neutron diffusion eigenvalue problem is theoretically analyzed. It is numerically and analytically proven that the convergence rate for CMFD2N algorithm is independent of the whole problem size but dependent on only the mesh size, and also unconditionally stable at least for the given problem. The convergence rate of the CMFD2N algorithm is explicitly calculated by Fourier analysis. From the form of the analytically calculated convergence rate, it is possible to know that the convergence rate is dependent on a product of the mesh size and the converged second harmonic buckling, which is defined by cross section data. The theoretically calculated convergence rate is identical with the numerically measured convergence rate. It is analytically proven that the convergence rate is unconditionally stable for the neutron diffusion eigenvalue problem. As further study, the new approach developed in this paper for the analysis of the convergence rate of CMFD2N algorithm will be applied to one-node CMFD eigenvalue problem.

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