Transactions of the Korean Nuclear Society Spring Meeting
Jeju, Korea, May 11-13, 2016

Application of Coarse Mesh Finite Difference Acceleration to Linear Discontinuous Transport Equation using Discrete Ordinate Method in Slab Geometry

Habib Muhammad, Ser Gi Hong*
Department of Nuclear Engineering, Kyung Hee University: 1732 Deogyeong-daero, Giheung-gu, Yongin, Gyeonggi-do, 446-701, Korea
**Corresponding author: sergihong@khu.ac.kr*

1. Introduction

The diffusion-like low-order equations have been used to significantly accelerate the solution to the neutron transport *k*-eigenvalue problem because the slow convergence of the power iteration is well-known for the reactor problems having large dominance ratio. Recently, the nonlinear diffusion acceleration NDA algorithm has been extended to solve fixed-source/*k*-eigenvalue problems with anisotropic scattering source and coarse mesh finite difference (CMFD) have been widely used in reactor physics area [1]. The objective of this paper is to apply the CMFD acceleration to the linear discontinuous discretization in the slab geometry and to investigate their effectiveness.

2. Methods and Results

2.1. Linear Discontinuous Spatial Discretization

The linear discontinuous (LD) method has become popular and extensively used in reactor physics and shielding analysis because of its accuracy and less susceptibility to the negative flux [3]. The transport equation for eigenvalue problem is given by

$$\mu_m \frac{\partial}{\partial x}\Psi_m^g(x) + \Sigma_t^g(x)\Psi_m^g(x) = q^g(x) \ , \tag{1}$$

$$q^g(x) = \Sigma_s^{g\to g}\phi^g + \frac{\chi_g}{k_{eff}}\sum_{g'=1}^{G}\nu\Sigma_f^{g'}\phi^{g'} + \sum_{g'\neq g}^{G}\Sigma_s^{g'\to g}\phi^{g'}. \tag{2}$$

In the LD method the flux and source are assumed to be approximated by linear functions that are discontinuous at the mesh edges. In the *i*'th cell, the flux is depicted as [3]:

$$\begin{aligned}\psi_{i,n}(x) &= \frac{1}{h_i}\left[\left(x_{i+1/2}-x\right)\psi_{i,n}^R + \left(x-x_{i-1/2}\right)\psi_{i,n}^L\right]\\ q^g(x) &= \frac{1}{h_i}\left[\left(x_{i+1/2}-x\right)q_i^R + \left(x-x_{i-1/2}\right)q_i^L\right]\end{aligned} \tag{3}$$

Here R and L refer to the right and left flux values for the *i*'th mesh. The discontinuity means that the angular flux at the right edge of the *i*'th mesh can be different from the one at the left edge of its adjacent (*i*+1)'th mesh. The cross sections are assumed to be constant for each mesh. Since there are two unknown fluxes per cell $\psi_{i,m}^R$ and $\psi_{i,m}^L$ two equations are required per cell. These equations are obtained by inserting Eq. (3) into Eq. (1) and integrating over the mesh cell after multiplying with the linear expansion functions. The resulting equations are

$$For\,\mu_m > 0$$

$$\psi_{i,m}^L = \frac{(\frac{1}{2}\mu_m + \frac{1}{3}h_i\sigma_t)\left[\mu_m\psi_{i-1,m}^R + \frac{1}{3}h_i q_{m,i}^L + \frac{1}{6}h_i q_{m,i}^R\right] - (\frac{1}{2}\mu_m + \frac{1}{6}h_i\sigma_t)\left[\frac{1}{6}h_i q_{m,i}^L + \frac{1}{3}h_i q_{m,i}^R\right]}{\frac{1}{2}\mu_m^2 + \frac{1}{12}h_i\sigma_t^2 + \frac{1}{3}\mu_m h_i\sigma_t}, \tag{4}$$

$$\psi_{i.m}^R = \frac{(\frac{1}{2}\mu_m - \frac{1}{6}h_i\sigma_t)\left[\mu_m\Psi_{i-1,m}^R + \frac{1}{3}h_i q_{m,i}^L + \frac{1}{6}h_i q_{m,i}^R\right] + (\frac{1}{2}\mu_m + \frac{1}{3}h_i\sigma_t)\left[\frac{1}{6}h_i q_{m,i}^L + \frac{1}{3}h_i q_{m,i}^R\right]}{\frac{1}{2}\mu_m^2 + \frac{1}{12}h_i\sigma_t^2 + \frac{1}{3}\mu_m h_i\sigma_t}.$$

where $\psi_{i-1,m}^R$ is the upstream angular flux which comes from the previous mesh calculation.

2.2 CMFD Acceleration

In this section, the CMFD equations for accelerating the power iteration of the LD method are derived.

The derivation starts with the following balance equation which is obtained by integrating Eq. (1) over angle [2].

$$\frac{dJ^g}{dx} + \Sigma_t^g\phi^g = \sum_{g'=1}^{G}\Sigma_{s,0}^{g'\to g}\phi^{g'} + \frac{\chi_g}{k_{eff}}\sum_{g'=1}^{G}\nu\Sigma_f^{g'}\phi^{g'}. \tag{5}$$

Then, the discretized form of the Eq. (5) is obtained by integrating it over a coarse mesh.

$$J_{ic+1/2}^g - J_{ic-1/2}^g + \Sigma_{r,ic}^g\bar{\phi}_{ic}^g h_{ic} = h_{ic}\bar{s}_{ic}^g. \tag{6}$$

where the index "*ic*" denotes the coarse mesh and $J_{ic+1/2}^g$ the net current at the right edge of the coarse mesh. The source is given by

$$\bar{s}_{ic}^g = \frac{\chi_g}{k_{eff}}\sum_{g'=1}^{G}\nu\Sigma_f^{g'}\bar{\phi}_{ic}^{g'} + \sum_{g'\neq g}^{G}\Sigma_s^{g'\to g}\bar{\phi}_{ic}^{g'}. \tag{7}$$

Then, we use the following equation for the relationship between the coarse mesh scalar flux and the interface net current:

$$J_{ic+1/2} = -\tilde{D}_{ic+1/2}(\bar{\phi}_{ic+1} - \bar{\phi}_{ic}) + \hat{D}_{ic+1/2}(\bar{\phi}_{ic+1} + \bar{\phi}_{ic}). \tag{8}$$

where

$$\tilde{D}_{ic+1/2}=\frac{2\beta_{ic}\beta_{ic+1}}{\beta_{ic}+\beta_{ic+1}} \text{ and } \beta_{ic}=\frac{1}{3h_{ic}\Sigma^g_{t,ic}}.$$

In Eq. (8), the correction factor $\hat{D}^g_{ic+1/2}$ should be determined to preserve the net current from the transport calculations. Then, the substitution of Eq. (8) into Eq. (6) gives the following CMFD equation.

$$\begin{aligned}&-(\tilde{D}_{ic-1/2}+\hat{D}^g_{ic-1/2})\bar{\phi}^g_{ic-1}+\left(\tilde{D}_{ic-1/2}+\hat{D}^g_{ic-1/2}+\tilde{D}_{ic+1/2}+\hat{D}^g_{ic+1/2}+h_{ic}\Sigma^g_r\right)\bar{\phi}^g_{ic}\\&-(\tilde{D}_{ic+1/2}+\hat{D}^g_{ic+1/2})\bar{\phi}^g_{ic+1}=h_{ic}\bar{s}^g_{ic}.\end{aligned} \quad (9)$$

The correction factor is determined by using the transport calculation results with the following equations:

$$\hat{D}^g_{ic+1/2}=\frac{\tilde{D}_{ic+1/2}\left(\bar{\phi}^H_{ic+1}-\bar{\phi}^H_{ic}\right)+J^g_{ic+1/2}}{\left(\bar{\phi}^H_{ic+1}+\bar{\phi}^H_{ic}\right)}. \quad (10)$$

where $\bar{\phi}^H_{ic}$ is the coarse mesh average flux obtained from the transport results. The interface net current for the correction factor in Eq. (10) should be carefully calculated by using the upstream partial currents as follows:

$$J^{R+}_{fr(ic)}=\frac{1}{2}\sum_{m=1}^{M/2}\mu_m w_m \psi^R_{fr(ic),m}, \quad (11)$$

$$J^{L-}_{fl(ic+1)}=\frac{1}{2}\sum_{m=M/2+1}^{M}|\mu_m| w_m \psi^L_{fl(ic+1),m}. \quad (12)$$

where $fr(ic)$ is the rightmost fine mesh contained in the ic'th coarse mesh and $fl(ic+1)$ is the leftmost fine mesh contained in the ic+1'th coarse mesh. In Eqs. (11) and (12), $J^{R+}_{fr(ic)}$ and $J^{L-}_{fl(ic+1)}$ are the right and left moving partial currents, respectively. Then, the partial current at the interface of the coarse mesh boundary is calculated as [4]:

$$J_{ic+1/2}=J^{R+}_{fr(ic)}-J^{L-}_{fl(ic+1)}. \quad (13)$$

Our observation showed that Eq. (13) is critical to achieve the rapid convergence of the CMFD acceleration of the LD discretization. On the other hand, we used slightly different equations using the partial currents for the correction factors at the boundaries from the one for the internal edges.

For example, the correction factors at the left boundary are given by:

$$\begin{aligned}\hat{D}^-_{1/2}&=\beta_1\frac{\phi_1}{\phi_{1/2}}-(\frac{1}{4}+\beta_1)+\frac{J^-_{1/2}}{\phi_{1/2}},\\ \hat{D}^+_{1/2}&=-\beta_1\frac{\phi_1}{\phi_{1/2}}-(\frac{1}{4}-\beta_1)+\frac{J^+_{1/2}}{\phi_{1/2}}.\end{aligned} \quad (14)$$

2.3 Coupling Procedure of Linear Discontinuous Transport Equation and CMFD

Now we describe here the coupling procedure of linear discontinuous transport equation and coarse mesh finite difference (CMFD) for acceleration. The procedure can be summarized as follows:

Step-I: Solve the CMFD equations (i.e., Eq. (9)) for flux $\bar{\phi}^l_{ic}$ and eigenvalue k^l_{ic} with the initial guess of the eigenvalue and coarse mesh flux or the previous iteration values. The correction factors for the first power iteration are set to zero.

Step-II: Set $k_{eff}=k_{ic}$ and it is used as the initial guess for the transport calculation. And the fine mesh scalar fluxes for the transport calculation are calculated by using the following prolongation:

$$\bar{\phi}^H_{ic}=\frac{\bar{\phi}^{H,l-1}_i}{\bar{\phi}^{H,l-1}_{ic}}\bar{\phi}^l_{ic}, i\in ic.$$

Step-III: Solve the LD equation Eq. (4) and the corresponding equations for the left moving direction to get $\bar{\phi}^{H,l}_i$, k^l_{eff} and $\bar{\phi}^{H,l}_{ic}$, $J^{H,l}_{ic+1/2}$. At present, only one energy group sweeping with 5 transport sweeps for each energy group is used in the transport calculations.

Step-IV: Update the correction factor $\hat{D}^g_{ic+1/2}$ using Eq. (10) and check the convergence of the fission source and the eigenvalue. If not converged, then go to Step-I.

3. Numerical Test

In this section, the CMFD acceleration coupled with the LD discretization is numerically tested to show the effectiveness. We considered two different slab reactor problems having different sizes with the same two-group cross sections. Figure 1 shows the configuration of the slab reactor eigenvalue problem. As shown in Figure 1, the reactor consists of the central core and outer reflector. The vacuum boundary conditions are imposed on the both boundaries. The core and the reflector thicknesses are 60cm and 12cm, respectively, for the first test problem while they are 300cm and 20cm for the second test problem. The second test problem was selected for showing the effectiveness of the CMFD acceleration for the large size problem. The two-group macroscopic cross sections are summarized in Table 1.

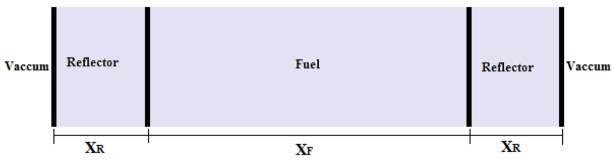

Figure 1: Configuration of the Two Group Eigenvalue Problem

Table 1: Cross Sections (cm^{-1}) For the Two Group Eigenvalue Problem

	Fuel		Reflector	
Energy Group	1	2	1	2
Total	2.23775E-01	1.03864E+00	2.50367E-01	1.64482E+00
$\nu\Sigma_f$	9.09319E-03	2.90183E-01	0.00000E+00	0.00000E+00
1→g	1.92423E-01	2.28253E-02	1.93446E-01	5.65042E-02
2→g	0.00000E+00	8.80439E-01	0.00000E+00	1.62452E+00

Table 2 summarizes the numbers of iterations for the first test problems for the different coarse mesh sizes while we fixed the fine mesh size to 1.0cm. The convergence criterions for the point-wise fission source and the eigenvalue are 10^{-5} and 10^{-7}, respectively. Table 2 shows the transport calculation without CMFD converges in 94 power iterations while the CMFD acceleration coupled with LD converges very rapidly and the number of iterations ranges from 11 to 45. In particular, it is noted that the number of iterations with CMFD increases as the coarse mesh size increases.

Table 2: Iteration Counts for Different Coarse Mesh Sizes for the First Test Problem
(X_F =60.0 cm, X_R =12.0 cm)

Coarse Mesh Size [cm]	Number of Power Iterations without CMFD	k_{eff}	Number of Power Iterations with CMFD	k_{eff}
1.0	94	1.50557	11	1.50557
2.0	94	1.50557	12	1.50557
3.0	94	1.50557	14	1.50557
4.0	94	1.50557	20	1.50557
6.0	94	1.50557	27	1.50557
12.0	94	1.50557	45	1.50557

Table 3 summarizes the results of the second test problem where the thicknesses of the core and reflector are 300cm and 20cm, respectively. Table 3 shows that the CMFD acceleration coupled with LD for this large size problem is more efficient than for the first test problem of the small size. As in Table 2, the number of iterations with CMFD increases as the coarse mesh size increases but there is one exception that the smallest coarse mesh size case of 1.0cm has the larger number of iterations than the cases having 2.0 and 4.0 cm thick coarse meshes. In Tables 2 and 3, it should be noted that the CMFD acceleration gave the exactly same k_{eff} value as the transport calculations with CMFD. Figure 2 compares the fast and thermal group scalar flux distributions obtained with and without the CMFD acceleration for the first test problems. As shown in the figure, the scalar fluxes obtained with the CMFD acceleration is exactly agreed with those obtained with the transport calculations without CMFD.

Table 3: Iteration Counts for Different Coarse Mesh Sizes for the Second Test Problem
(X_F =300.0 cm, X_R =20.0 cm)

Coarse Mesh Size [cm]	Number of Power Iterations without CMFD	k_{eff}	Number of Power Iterations with CMFD	k_{eff}
1.0	491	1.61769	24	1.61769
2.0	491	1.61769	10	1.61769
4.0	491	1.61769	20	1.61769
5.0	491	1.61769	25	1.61769
10.0	491	1.61769	42	1.61769
20.0	491	1.61769	76	1.61769

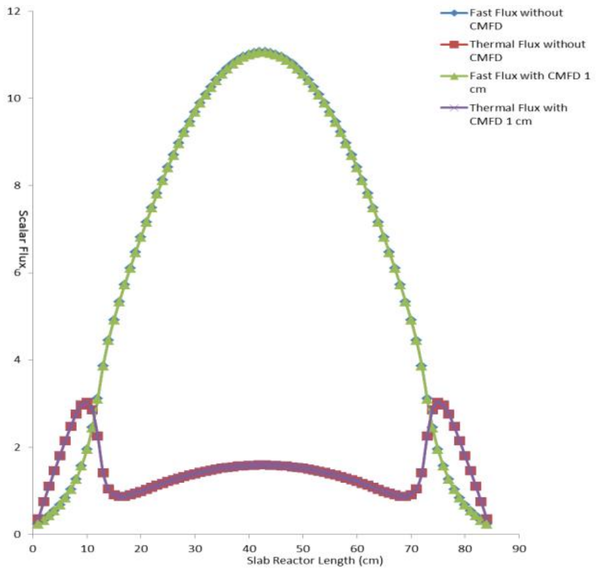

Figure 2: Scalar Flux Distribution for Test Problem 1

4. Summary and Conclusion

In this work, the CMFD acceleration method was applied to the LD discretization for the discrete ordinates transport equation in slab geometry and its effectiveness was investigated with the numerical tests for two different size slab reactor problems. In particular, the rapid convergence of CMFD was achieved with the correction factors which were calculated with the interface net currents in terms of the upstream partial currents. The results of the numerical tests showed that our CMFD coupled with LD significantly accelerated the power iteration of the transport calculations both for the small and large size problems and the convergence was slowly degraded as the coarse mesh size increases. So, it can be concluded that the linear discontinuous transport method with the coarse mesh finite difference (CMFD) is very efficient and helpful to solve the reactor problems.

REFERENCES

[1] J. Willert, H. Park and W. Taitano, "Applying Nonlinear Acceleration to the Neutron Transport *k*-Eigenvalue Problem with Anisotropic Scattering", *Nucl. Sci. Eng*., **181**, 1-10 (2015).

[2] J. Willert, H. Park and W. Taitano, "Using Anderson Acceleration to Accelerate the Convergence of Neutron Transport Calculations with Anisotropic Scattering", *Nucl. Sci. Eng*., **181**, 342-351 (2015).

[3] E. E. Lewis and W. F. Miller, "*Computational Methods of Neutron Transport*", John Wiley, New York (1984).

[4] Dong Wook Lee and Han Gyu Joo, "Coarse Mesh Finite Difference Acceleration of Discrete Ordinate Neutron Transport Calculation Employing Discontinuous Finite Element Method", *Dept of Nucl Engg*, Seoul National University, Jan 2014.