Extension of NPMM for Higher Mode Solutions

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1. Introduction

Previous studies show that the Noise Propagation Matrix Method (NPMM) can be used to compute the dominance ratio of the system [1]. The NPMM is essentially the same as the Coarse Mesh Projection Method (CMPM) proposed even earlier [2-8], both of which use the noise propagation matrix (NPM) based on a coarse mesh to determine the dominance ratio. The eigenvalues and eigenfunctions of the NPM are the eigenvalue ratios and the eigenfunctions of the system.

Recently the Modified Power Method (MPM) has been studied to calculate the higher mode eigensolutions using the transfer matrix (TM), whose eigenvalues and eigenvectors are the eigenvalues and eigenfunctions of the system [9, 10].

In NPMM, only the fundamental fission sources are utilized, and the higher mode information is from the statistical noise of the fundamental fission sources. However, in MPM, the higher modes are explicitly simulated at the same time as the fundamental fission source. Due to the similarity of the NPM and TM, it is natural to extend the NPMM to get higher mode eigensolutions by simulating higher mode fission sources explicitly.

2. Methods and Results

In this section the basic theory of the NPMM will be presented, followed by the extension and numerical tests.

2.1 The Noise Propagation Matrix

Both the CMPM and the NPMM use a NPM to compute the dominance ratio, which is calculated from two source correlation matrices. The fission source is discretized on a small number of large meshes. Let $\mathbf{s}^{(m)}$ denote the fission source vector at the end of cycle *m*, and assume that *m* is sufficiently large that the source vector can be considered a stationary random variable. Following Nease and Ueki, the fission source vector can be decomposed as:

$$\mathbf{s}^{(m)} = N\mathbf{s}_0 + \sqrt{N}\mathbf{e}^{(m)}, \qquad (1)$$

where N is the number of histories per cycle, $\mathbf{s}_0$ is the normalized fundamental fission source vector, and $\mathbf{e}^{(m)}$ is a normalized stochastic noise vector representing the derivation of the cycle-*m* fission source from its expected value. Denoting the ensemble-averaging process by $\langle\cdots\rangle$, there is:

$$\left\langle \mathbf{s}^{(m)} \right\rangle = N\mathbf{s}_0 \equiv \mathbf{s}. \qquad (2)$$

From Eqs. (1) and (2), there is:

$$\left\langle \mathbf{e}^{(m)} \right\rangle = 0. \qquad (3)$$

Nease and Ueki show that $\mathbf{e}^{(m)}$ is propagated from cycle to cycle according to:

$$\mathbf{e}^{(m+1)} = \mathbf{A}_0\mathbf{e}^{(m)} + \boldsymbol{\varepsilon}^{(m+1)}, \qquad (4)$$

where $\mathbf{A}_0$ is the NPM and $\boldsymbol{\varepsilon}^{(m+1)}$ is a vector representing the noise during the neutron transport process at cycle-(m+1). The latter has the property:

$$\left\langle \boldsymbol{\varepsilon}^{(m+1)} \right\rangle = 0. \qquad (5)$$

Although the NPM is defined based on the noises, the practical implementations of the CMPM and NPMM have used an alternative way utilizing the correlation matrices based on the fission sources.

Based on the Eqs. (1) and (4), and using the property $\mathbf{A}_0\mathbf{s}_0 = 0$, it can be shown that the equation governing the propagation of the fission source from cycle to cycle is:

$$\mathbf{s}^{(m+1)} = \mathbf{A}_0\mathbf{s}^{(m)} + \boldsymbol{\eta}^{(m+1)}, \qquad (6)$$

where $\boldsymbol{\eta}^{(m)} = N\mathbf{s}_0 + \sqrt{N}\boldsymbol{\varepsilon}^{(m)}$. Multiplying Eq. (6) with $\mathbf{s}^{(m)\mathrm{T}}$ from right and ensemble-averaging results in:

$$\left\langle \mathbf{s}^{(m+1)}\mathbf{s}^{(m)T} \right\rangle = \mathbf{A}_0 \left\langle \mathbf{s}^{(m)}\mathbf{s}^{(m)T} \right\rangle + \left\langle \boldsymbol{\eta}^{(m+1)}\mathbf{s}^{(m)T} \right\rangle, \qquad (7)$$

where

$$\left\langle \boldsymbol{\eta}^{(m+1)}\mathbf{s}^{(m)T} \right\rangle = \left\langle \left(N\mathbf{s}_0 + \sqrt{N}\boldsymbol{\varepsilon}^{(m+1)}\right)\left(N\mathbf{s}_0 + \sqrt{N}\mathbf{e}^{(m+1)}\right)^T \right\rangle. \qquad (8)$$

The noise introduced in one cycle is uncorrelated to the accumulated noise terms from all previous cycles, i.e. :

$$\left\langle \boldsymbol{\varepsilon}^{(m+1)} \mathbf{e}^{(m)T} \right\rangle = 0. \quad (9)$$

Using Eqs. (2), (3), (5) and (9), Eq. (8) is:

$$\left\langle \boldsymbol{\eta}^{(m+1)} \mathbf{s}^{(m)T} \right\rangle = \mathbf{s}\mathbf{s}^T. \quad (10)$$

Define the source correlation matrices as:

$$\mathbf{L}_0' \equiv \left\langle \mathbf{s}^{(m)} \mathbf{s}^{(m)T} \right\rangle, \ \mathbf{L}_1' \equiv \left\langle \mathbf{s}^{(m+1)} \mathbf{s}^{(m)T} \right\rangle. \quad (11)$$

From Eqs. (7), (10) and (11), the NPM can be obtained:

$$\mathbf{A}_0 = \left[\mathbf{L}_1' - \mathbf{s}\mathbf{s}^T \right] \left(\mathbf{L}_0' \right)^{-1}. \quad (12)$$

The largest-modulus eigenvalue of the NPM is the dominance ratio of the system.

2.2 Extension of the NPM to higher modes

Similar to Eq. (1), the stationary fundamental fission source at the beginning of cycle-*m* can be described as:

$$\mathbf{v}_0^{(m)} = N\mathbf{v}_0 + \sqrt{N}\mathbf{e}_{00}^{(m)}, \quad (13)$$

while the corresponding fundamental fission produced neutron distribution at the end of cycle-*m* is:

$$\mathbf{w}_0^{(m)} = k_0 N\mathbf{v}_0 + k_0\sqrt{N}\mathbf{e}_{01}^{(m)}, \quad (14)$$

where $\mathbf{v}_0^{(m)}$ and $\mathbf{w}_0^{(m)}$ are the fundamental fission neutron distributions at the beginning and end of cycle-*m*, $\mathbf{v}_0$ is the normalized fundamental fission source, N is the number of neutron histories per cycle, k_0 is the fundamental mode eigenvalue, $\mathbf{e}_{00}^{(m)}$ and $\mathbf{e}_{01}^{(m)}$ are the normalized stochastic noise vectors that represent the deviation of fundamental mode fission neutron distributions to their expected values at the beginning and end of cycle-*m*, respectively.

Similar to Eq. (4), the noise propagation equation can be written as:

$$k_0 \mathbf{e}_{01}^{(m)} = \mathbf{P}\mathbf{e}_{00}^{(m)} + k_0 \boldsymbol{\varepsilon}_0^{(m)}, \quad (15)$$

where **P** can be treated as the NPM but with some modification, compared with Eq. (4), and it is essentially the same as TM, as will be revealed in the following discussions. $\boldsymbol{\varepsilon}_0^{(m)}$ is the stochastic noise introduced to the fundamental mode neutron distribution at cycle-*m*. According to Eqs. (14) and (15) and using the property $\mathbf{P}\mathbf{v}_0 = k_0 \mathbf{v}_0$, it can be derived that:

$$\mathbf{w}_0^{(m)} = \mathbf{P}\mathbf{v}_0^{(m)} + k_0\sqrt{N}\boldsymbol{\varepsilon}_0^{(m)}. \quad (16)$$

Similar to Eq. (16), if the *i*-th mode neutron sources can be simulated, the *i*-th mode fission neutron distributions at the beginning and end of cycle-*m* can be related with the following equation:

$$\mathbf{w}_i^{(m)} = \mathbf{P}\mathbf{v}_i^{(m)} + k_i\sqrt{N}\boldsymbol{\varepsilon}_i^{(m)}, \quad (17)$$

where $\mathbf{v}_i^{(m)}$ and $\mathbf{w}_i^{(m)}$ are the *i*-th mode fission neutron distributions at the beginning and end of cycle-*m*, k_i is the *i*-th mode eigenvalue and $\boldsymbol{\varepsilon}_i^{(m)}$ is the stochastic noise introduced to *i*-th mode neutron source during cycle-*m*.

According to Eqs. (16) and (17), the relation between the first N eigenmode fission sources at the beginning and end of cycle-m can be expressed as:

$$\mathbf{W}^{(m)} = \mathbf{P}\mathbf{V}^{(m)} + \mathbf{U}^{(m)}, \quad (18)$$

where the columns of $\mathbf{V}^{(m)}$ and $\mathbf{W}^{(m)}$ represent the first N eigenmode neutron sources at the beginning and end of cycle-*m*, respectively, the columns of $\mathbf{U}^{(m)}$ represents the stochastic noises introduced to the first N eigenmode fission sources during cycle-*m*, $\mathbf{W}^{(m)}, \mathbf{V}^{(m)}, \mathbf{U}^{(m)} \in \mathbf{R}_{M\times N}$, $\mathbf{P} \in \mathbf{R}_{M\times M}$, and M is number of meshes that used to discretize the neutron sources. Using the property $\left\langle \mathbf{U}^{(m)} \right\rangle = 0$, it can be derived from Eq. (18) that:

$$\left\langle \mathbf{W}^{(m)} \right\rangle = \mathbf{P} \left\langle \mathbf{V}^{(m)} \right\rangle. \quad (19)$$

Since the stochastic noises introduced during cycle-*m* are independent of the neutron sources, from Eqs. (9) and (13) it can be derived that:

$$\left\langle \mathbf{U}^{(m)} \mathbf{V}^{(m)T} \right\rangle = \mathbf{0}. \quad (20)$$

Multiplying Eq. (18) with $\mathbf{V}^{(m)T}$ from right and ensemble-averaging results in:

$$\left\langle \mathbf{W}^{(m)} \mathbf{V}^{(m)T} \right\rangle = \mathbf{P} \left\langle \mathbf{V}^{(m)} \mathbf{V}^{(m)T} \right\rangle. \quad (21)$$

The TM (modified NPM) can then be calculated as:

$$\mathbf{P} = \left\langle \mathbf{W}^{(m)} \mathbf{V}^{(m)T} \right\rangle \left\langle \mathbf{V}^{(m)} \mathbf{V}^{(m)T} \right\rangle^{-1}. \quad (22)$$

It should be noticed that the eigenvalues of the TM are the eigenvalues of the system, and the eigenvectors of the TM are the fission sources represented with the M coarse meshes, as already well explained in the MPM.

2.3 The implementation of the extended NPMM

The implementation of the extended NPMM is very similar to that of the MPM, except for the calculation method of the TM. The correlation matrices are accumulated:

$$\tilde{\mathbf{L}}_0^{(m)} \equiv \sum_{i=m_0+1}^{m} \mathbf{V}^{(i)}\mathbf{V}^{(i)T}, \tag{23}$$

$$\tilde{\mathbf{L}}_1^{(m)} \equiv \sum_{i=m_0+1}^{m} \mathbf{W}^{(i)}\mathbf{V}^{(i)T}, \tag{24}$$

where m_0 is the number of cycles skipped before accumulation of the correlation matrices. The cycle-m estimate of the TM is then computed as:

$$\mathbf{P}^{(m)} = \tilde{\mathbf{L}}_1^{(m)}\left(\tilde{\mathbf{L}}_0^{(m)}\right)^{-1}. \tag{25}$$

The first N eigenvectors of the TM are used to calculate the combination coefficient matrix that is then used to update the neutron sources, as already described in the MPM.

2.4 The 2D square neutron transport problem

The multi-group 2D homogeneous square neutron transport problem was modeled to demonstrate the extended NPMM (ENPMM). The 7-group cross sections are from the C5G7 benchmark for the 'mox8' material. The side length of the 2D square is 400 cm, with black boundary on four sides. The 5x5 uniform coarse meshes were used to discretize the fission sources and obtain the correlation matrices, while 8 or 16 eigenmodes were simulated at the same time. The Monte Carlo simulations were done with 400 inactive cycles, 400 active cycles and 500,000 histories per cycle.

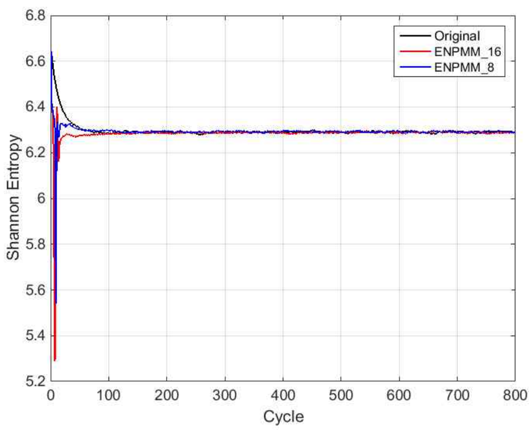

Fig. 1. The Shannon Entropy results of different methods.

The Shannon Entropy results of different methods are shown in Fig. 1. It can be seen that the results are consistent for all three simulations, and generally with more modes, the convergence of ENPMM is quicker.

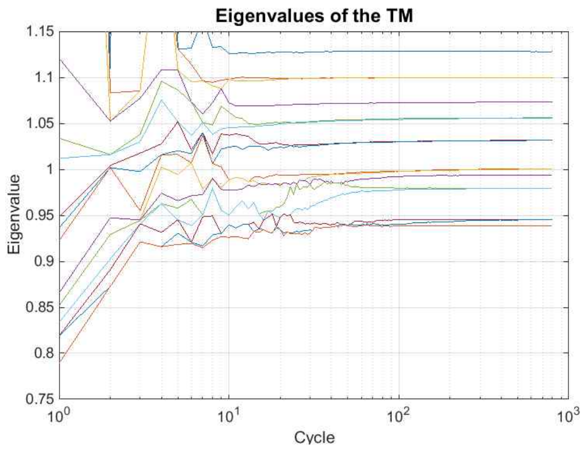

Fig. 2. Eigenvalues of the TM that is calculated at evry cycle.

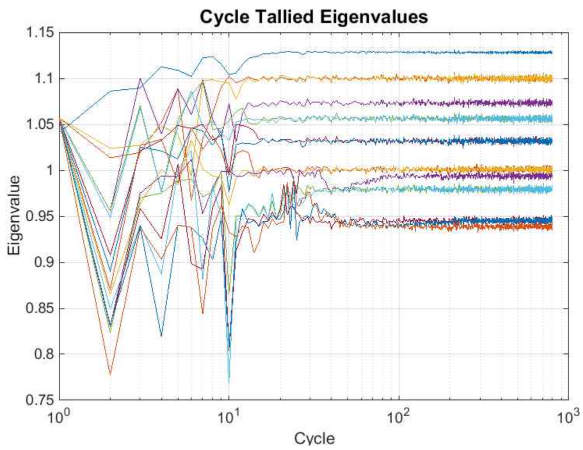

Fig. 3. Eigenvalues tallied at every cycle as the ratio of the total absolute weight values to the number of histories per cycle.

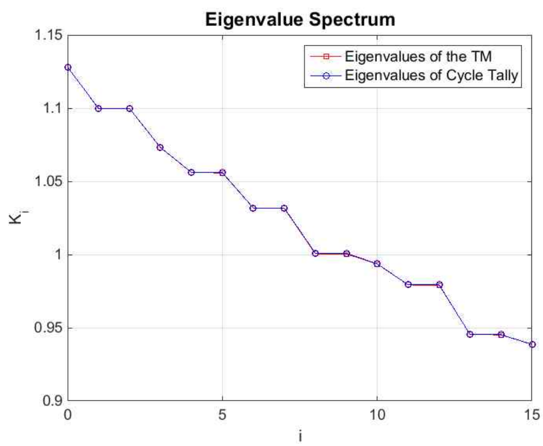

Fig. 4. The eigenvalue spectrum.

The eigenvalue results of ENPMM_16 are shown in Figs. 2-4. It can be seen that at the beginning of the simulation there is very big fluctuations, which is thought to be caused by the less fission source pairs being used to calculate the correlation matrices.

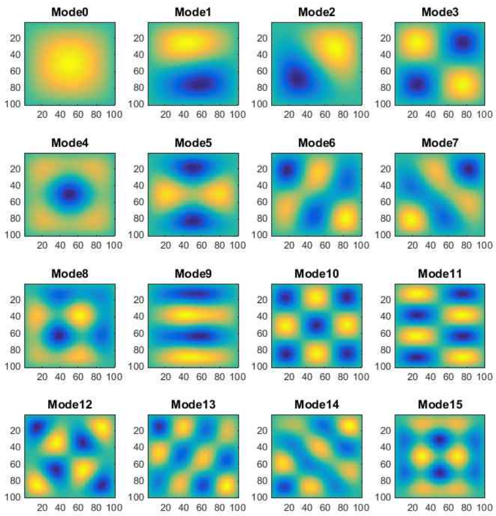

Fig. 5. The fissin source distrubutions of the first 16 eigenmodes.

The first 16 eigenmode fission source distribution obtained with ENPMM_16 are shown in Fig. 5, with 100x100 uniform meshes to discretize the fission sources.

3. Conclusions

The NPMM has been extended to calculate the higher mode solutions with the Monte Carlo simulation. Previously the NPMM was used to calculate the dominance ratio based on the stochastic noise contained in the fundamental fission source and large number of active cycles is required to get accurate results. The NPMM was combined with the MPM with the higher mode fission sources simulated explicitly in this study, and various higher mode solutions can be accurately calculated with less number of active cycles.

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