Development of Ellipse Equation for Spray Droplet Shape Modeling

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1. Introduction

A water droplet of a containment spray system is similar to elliptical objects. According to Clift's experimental study, these water droplets are non-spherical in falling mechanics. The fission products removal mechanism of water droplets depends on the nature of flow around the droplets. In these flow regimes, the flatness of the droplets strongly affects the fission products removal process. In this work, the flatness of water droplets is efficiently calculated using a new methodology based on developed ellipse equations. The mathematical technique in this work is based on ellipse integral and ellipse geometric equations.

2. Methodology

In this section, some fundamental forms are introduced to develop new methods to illustrate the nature of spray droplet shape.

2.1 Flatness of spray droplet

The flatness of droplets depends on the eccentricity e. Here, the pattern of eccentricity is calculated using an incomplete ellipse integral of the first kind as follows:

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin\theta}} \tag{1}$$

where k^2 is the elliptic module (or ellipse parameter: $0< k^2 <1$). Here, letting $t=\sin\theta$ and $dt=\cos\theta\, d\theta = \sqrt{1-t^2}d\theta$, equation (1) can be written as

$$F(\varphi, k) = \int_0^{\sin\varphi} \frac{dt}{\sqrt{(1-k^2t^2)(1-t^2)}} \tag{2}$$

Letting $u=\tan\theta$ and $du = \sec^2\theta\, d\theta = (1-u^2)d\theta$, then equation (2) can also be written as

$$F(\varphi, k) = \int_0^{\tan\varphi} \frac{du}{\sqrt{(1-u^2)(1-k'u^2)}} \tag{3}$$

where $k' = 1-k^2$ is the complementary elliptic module.
The inverse function of $F(\varphi,k)$ is given by the Jacobi amplitude, expressed as $F^{-1}(m,k)=\varphi=am(m,k)$.
The integral is given by

$$I = \frac{1}{\sqrt{2}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \tag{4}$$

This is an elliptic integral of the first kind.

Here, we use $\cos\theta = 1-2\sin^2\left(\frac{1}{2}\theta\right)$ and $\sin\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1-\cos\theta}{2}}$, and then rewrite

$$\sqrt{\cos\theta - \cos\theta_0} = \sqrt{1-2\sin^2\left(\frac{1}{2}\theta\right) - \cos\theta_0} \tag{5}$$

$$= \sqrt{1-\cos\theta_0}\sqrt{1-\frac{2}{1-\cos\theta_0}2\sin^2\left(\frac{1}{2}\theta\right)} \tag{6}$$

so

$$I = \frac{1}{\sqrt{2}} \int_0^{\theta_0} \frac{d\theta}{\sin\left(\frac{1}{2}\theta\right)\sqrt{1-2\csc^2\left(\frac{1}{2}\theta_0\right)\sin^2\left(\frac{1}{2}\theta\right)}} \tag{7}$$

Now, let $\sin\left(\frac{1}{2}\theta\right) = \sin\left(\frac{1}{2}\theta_0\right)\sin\varphi$
The angle θ is thus transformed to

$$\varphi = \sin^{-1}\left[\frac{\sin\left(\frac{1}{2}\theta\right)}{\sin\left(\frac{1}{2}\theta_0\right)}\right] \tag{8}$$

which ranges from 0 to $\pi/2$ as θ varies from 0 to θ_0.
Taking the differential form

$$\frac{1}{2}\cos\left(\frac{1}{2}\theta\right) d\theta = \sin\left(\frac{1}{2}\theta_0\right)\cos\varphi d\varphi, \tag{9}$$

equation (9) can be changed to equation (10)

$$\frac{1}{2}\sqrt{1-2\sin^2\left(\frac{1}{2}\theta_0\right)\sin^2\varphi}\, d\theta \tag{10}$$

Applying equations (9) and (10) into equation (7), equation (7) is rewritten as follows:

$$I = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2\left(\frac{1}{2}\theta_0\right)\sin^2\varphi}} \frac{\sin\left(\frac{1}{2}\theta_0\right)\cos\varphi d\varphi}{\sin\left(\frac{1}{2}\theta_0\right)\sqrt{1-\sin^2\varphi}} \tag{11}$$

Hence, equation (11) is changed into equation (12).

$$= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\sin^2\left(\frac{1}{2}\theta_0\right)\sin^2\varphi}} \tag{12}$$

Here, equation (12) has the same form as equation (1). These forms can be written as follows:

$$F(\varphi, k) = F\left(\frac{\pi}{2}, e\right) = F\left(\frac{\pi}{2}, \sin\left(\frac{1}{2}\theta_0\right)\right) \tag{13}$$

The flatness of spray droplets, eccentricity e, is then selected by the form of $\sin\left(\frac{1}{2}\theta_0\right)$, and θ_0 ranges from 0 to $\frac{\pi}{2}$.

2.2 Conditions of Ellipse

The general form of the ellipse is delineated in equation (14).

$$a\,x^2 + 2b\,xy + c\,y^2 + 2\,dx + 2\,fy + g = 0 \tag{14}$$

This is an ellipse after satisfying the conditions of (15) and (16) as below:

$$\Delta= \begin{vmatrix} a & b & d \\ d & c & f \\ d & f & g \end{vmatrix},\ J = \begin{vmatrix} a & b \\ b & c \end{vmatrix},\ I = a + c \qquad (15)$$

$$\Delta \neq 0,\ \ J > 0,\ \frac{\Delta}{I} < 0,\ a \neq c,\ J = ac - b^2 \neq 0 \quad (16)$$

In this case, the center of the ellipse (x_0, y_0) is given by

$$x_0 = \frac{cd-bf}{b^2-ac} \qquad y_0 = \frac{af-bd}{b^2-ac} \qquad (17)$$

The lengths of the semi-axes are as given below:

$$a' = \sqrt{\frac{2(af^2+cd^2+gb^2-2bdf-acg)}{(b^2-ac)\left[\sqrt{(a-c)^2+4b^2}-(a+c)\right]}} \qquad (18)$$

$$b' = \sqrt{\frac{2(af^2+cd^2+gb^2-2bdf-acg)}{(b^2-ac)\left[-\sqrt{(a-c)^2+4b^2}-(a+c)\right]}} \qquad (19)$$

where a, b, c, d, e, f, and g are random variables ranging from 0 to 1.

2.3 Conditions of Shape

From the conditions of section 2.1 and section 2.2, the shape of spray droplets is derived as follows:
If r and θ are measured from a focus F instead of from the center C of (17), then the equation of the ellipse is

$$X=C + r\cos\theta \quad Y= r\sin\theta \qquad (20)$$

This becomes

$$\frac{(C + r\cos\theta)^2}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1 \qquad (21)$$

Clearing the denominators, equation (21) gives

$$b^2(C^2 + 2C\, r\cos\theta + r^2\cos^2\theta) + a^2r^2\sin^2\theta = a^2b^2 \quad (22)$$

Applying the term $\sin^2\theta = 1 - \cos^2\theta$ and eccentricity e, equation (22) is changed into equation (23):

$$a^2(1-e^2)a^2e^2 + 2aea^2(1-e^2)r\cos\theta + a^2(1-e^2)r^2\cos^2\theta) + a^2r^2\cos^2\theta + a^2r^2 = a^2(1-e^2) \quad (23)$$

Dividing by the term $-a^2$ and simplifying equation (23), we obtain

$$-r^2 + [e\, r\cos\theta - a(1-e^2)]^2 = 0 \qquad (24)$$

This can be solved for r as given below:

$$r = \pm[e\, r\cos\theta - a(1-e^2)] \qquad (25)$$

For the conditions of the ellipse, it must take a negative sign, and hence becomes

$$r = a(1-e^2) - e\, r\cos\theta \qquad (26)$$

Simplifying for r,

$$r = \frac{a(1-e^2)}{1+ e\cos\theta} \qquad (27)$$

Here, $\theta = \frac{\pi}{2}$, r depends on a and e.

3. Results and Discussion

Using the ellipse equation, a new method to calculate the shape of spray droplet is introduced. Fig. 1 shows a comparison between the experimental results of another study and this calculation in terms of eccentricity. The results are in good agreement with the trend of the graph. Fig. 2 shows the spherical volume ratio of the droplets. In Fig. 2, the difference between the experimental results and this calculation is within 1%.

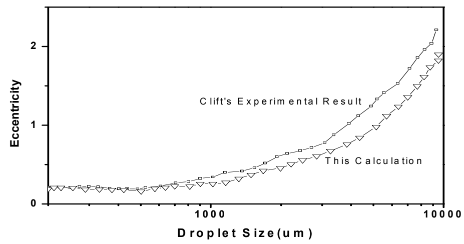

Fig. 1 Eccentricity compared with a previous study

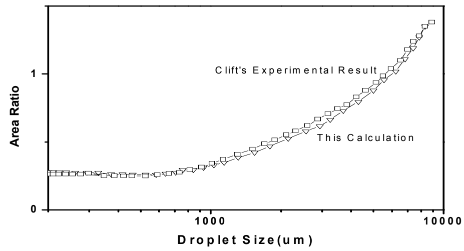

Fig. 2 Spherical volume ratio compared with a previous study

4. Conclusions

A new method to calculate the shape of spray droplets is developed using ellipse equations. All conditions of selecting the shape of spray droplets are developed. The application results are in good agreement with the graphical trend. The calculations of the spherical volume ratio are within error of 1% and have good identity compared with the experimental results. This method can be applied to predict the shape of water droplets without experimental effort.

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