SPH Simulation for Pinch Plasma using resistive MHD Model

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1. Introduction

A pinch is the phenomenon that appears in plasma when it is compressed by magnetic forces. In recent years, the pinch plasma has received large attention as an efficient source of radiation and a way to explore high-density plasma physics [1-3]. However, the experimental implementation of pinch plasma is difficult because it requires high-performance current sources and various diagnostic equipment. For this reason, various numerical approaches have been attempted to analyze pinch plasma, and the magnetohydrodynamic (MHD) simulation is one of the powerful tools for understanding it. Various MHD models are utilized depending on the type of plasma and applied assumption. In this case, it is appropriate to use the resistive MHD model since the pinch plasmas have varying local resistivity with temperature and pressure.

In this study, the non-ideal MHD model has been developed and implemented to the smoothed particle hydrodynamics (SPH) framework. SPH has many advantages, particularly in pinch simulations, as it imposes no restrictions on the symmetry of the problem to be solved and therefore handles complex physics relatively easily. This model is based on the existing SPH-based hydrodynamics code, but it includes some additional calculations essential for analyzing pinch plasma such as the magnetic pressure force and the resistive induction equation. In addition, this model also incorporates a variety of numerical techniques for capturing shock or reducing numerical instability. Finally, the simulations are conducted for some benchmark problems to verify the implemented SPH code.

2. Magnetohydrodynamics

Magnetohydrodynamics (MHD) is the study for the magnetic properties and behavior of electrically conducting fluids called plasma. Plasma is a state of matter composed of charged particles such as ions and electrons. In principle, in order to describe plasma behavior, the equation of motion of each particle should be calculated. However, it is practically impossible to solve the equations of motion for all particles that make up the plasma. Therefore, MHD equations are derived under the assumption that plasma can be considered as a single flow having the same density as ions [4]. The set of MHD equations are the combination of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electro-magnetism. These differential equations must be solved simultaneously, either analytically or numerically. In this case, various MHD equations can be derived depending on the type of plasma and applied assumption. The MHD equations used in this study are ideal-MHD based equations and resistive MHD equations that include the effect of plasma resistivity in ideal-MHD. In this section, we describe each governing equation constituting these MHD equations.

2.1 Ideal-MHD Governing Equations

The simplest form of MHD, Ideal MHD, assumes that the fluid has so little resistivity that it can be treated as a perfect conductor. In addition, in ideal-MHD, a single-fluid approximation that does not distinguish between electrons and ions is applied, and various physical quantities such as displacement current, electrical resistivity, viscosity, and thermal conduction are neglected. The ideal MHD equations consist of the continuity equation (Eq.(1)), the Cauchy momentum equation (Eq.(2)), the induction equation calculating the change in the magnetic field (Eq.(3)), and the energy conservation equation (Eq.(4)). Finally, the governing equations are closed by the equation of state (EOS) (Eq.(5)). These equations are expressed as follows [5]:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \boldsymbol{v}) = 0 \quad (1)$$

$$\frac{D\boldsymbol{v}}{Dt} = \frac{1}{\rho}\nabla \cdot \left(\frac{\boldsymbol{BB}}{\mu_0} - \left(\frac{1}{2\mu_0}\boldsymbol{B}^2 + P\right)\overleftrightarrow{I}\right) \quad (2)$$

$$\frac{D\boldsymbol{B}}{Dt} = -\boldsymbol{B}(\nabla \cdot \boldsymbol{v}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} \quad (3)$$

$$\frac{Du}{Dt} = -\frac{P}{\rho}\nabla \cdot \boldsymbol{v} \quad (4)$$

$$P = (\gamma - 1)\rho u \quad (5)$$

where $\rho, \boldsymbol{v}, P, \boldsymbol{B}$, and u are the mass density, the velocity, the thermal pressure, the magnetic field, and the internal energy, respectively, $\overleftrightarrow{I}$ is a unit tensor. In addition, γ and μ_0 are the ratio of specific heats for an adiabatic equation of state and permeability, respectively.

2.2 Resistive MHD Governing Equations

When the fluid cannot be considered as completely conductive, but the other conditions for ideal MHD are satisfied, it is possible to use an extended model called resistive MHD. In this model, the resistive term (Eq.(6), (7)) is added to the induction equation and energy equation of the ideal MHD model, and additional calculations are performed to obtain the current density (Eq.(8)). These added terms are expressed as follows [6]:

$$\left(\frac{D\boldsymbol{B}}{Dt}\right)_{resistive} = -\eta\nabla \times \boldsymbol{J} \quad (6)$$

$$\left(\frac{D\boldsymbol{u}}{Dt}\right)_{resistive} = \frac{1}{\rho}\eta \boldsymbol{J}^2 \quad (7)$$

$$\boldsymbol{J} = \frac{1}{\mu_0}\nabla \times \boldsymbol{B} \quad (8)$$

where, η is the plasma resistivity, and $\boldsymbol{J}$ is the current density, calculated as the curl of the magnetic field according to Ampere's law.

3. Smoothed Particle Hydrodynamics

Smoothed Particle Hydrodynamics (SPH) is a Lagrangian based particle method to solve fluid dynamics equations. Recently, it has been used in various fields with the development of computing techniques. The SPH method has definite advantages over the traditional grid-based numerical methods in dealing with applications that involve large deformations. For this reason, it is relatively easy to implement various types of physics, and therefore it is expected to be fit well into the simulation of pinch plasma. In this section, the basic concept of SPH method and the SPH formulations of the MHD model used to simulate plasma behavior are explained.

3.1 SPH Kernel Approximation

In the SPH method, the entire fluid system is expressed by a finite number of particles representing the material properties of that space, and the physical quantities such as density, momentum, and internal energy are calculated through the smoothing of neighboring particles. The smoothing procedure in the SPH method is based on the theory of integral interpolants using a delta function. However, the delta function is a discontinuous function, and hence it is difficult to handle numerically. To solve this problem, we can generalize the delta function to a continuous function W (known as the smoothing kernel) with a characteristic width h (known as the smoothing length), and the integral interpolant of a function f is defined as follow [7]:

$$< f(r) > = \int_{\Omega} f(r')W(r - r', h)d\Omega \quad (9)$$

The integral form of Eq. (9) can be discretized by representing the integral with a summation expression.

$$< f(\boldsymbol{r}_i) > = \sum_j f_j W(r_i - r_j, h)V_j \quad (10)$$

where $f(\boldsymbol{r}_i)$ is a function at the position $\boldsymbol{r}_i$, subscript j is the nearby particles of center particle i, and $V(= m/\rho)$ is the particle volume. Fig.1 shows the particle distribution with the kernel function. The value of the kernel weighting function is determined by the distance between particles, and it must be normalized over the entire computational domain.

In a similar way, spatial derivatives of a function can be also simply approximated by taking derivatives of a kernel function. In this case, the gradient (Eq.(11)), divergence (Eq.(12)), and curl (Eq.(13)) of the function expressed as follows, respectively [8]:

$$< \nabla f(\boldsymbol{r}_i) > = \sum_j f_j \nabla W(r_i - r_j, h)\frac{m_j}{\rho_j} \quad (11)$$

$$< \nabla \cdot f(\boldsymbol{r}_i) > = \sum_j f_j \cdot \nabla W(r_i - r_j, h)\frac{m_j}{\rho_j} \quad (12)$$

$$< \nabla \times f(\boldsymbol{r}_i) > = \sum_j f_j \times \nabla W(r_i - r_j, h)\frac{m_j}{\rho_j} \quad (13)$$

3.2 Resistive-MHD SPH Formulations

Based on the above SPH formula, the resistive MHD governing equations (described in Section 2.2) are expressed in the SPH code as:

$$\frac{d\rho_i}{dt} = \sum_j m_j \boldsymbol{v_{ij}} \cdot \nabla_i W_{ij} \quad (14)$$

$$\frac{d\boldsymbol{v_i}}{dt} = \sum_j m_j \left(\frac{\overleftrightarrow{\boldsymbol{M}_i}}{\rho_i^2} + \frac{\overleftrightarrow{\boldsymbol{M}_j}}{\rho_j^2}\right) \cdot \nabla_i W_{ij} \quad (15)$$

$$\boldsymbol{J}_i = -\frac{\rho_i}{\mu_0}\sum_j m_j \left(\frac{\boldsymbol{B}_i}{\rho_i^2} + \frac{\boldsymbol{B}_j}{\rho_j^2}\right) \times \nabla_i W_{ij} \quad (16)$$

$$\frac{d\boldsymbol{B}_i}{dt} = \frac{1}{\rho_i}\sum_j m_j (\boldsymbol{B}_i \boldsymbol{v}_{ij} - \boldsymbol{v}_{ij}\boldsymbol{B}_i) \cdot \nabla_i W_{ij} + \rho_i \sum_j m_j \left(\frac{\eta_i \boldsymbol{J}_i}{\rho_i^2} + \frac{\eta_j \boldsymbol{J}_j}{\rho_j^2}\right) \times \nabla_i W_{ij} \quad (17)$$

$$\frac{du_i}{dt} = \frac{1}{2}\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_i}{\rho_j^2}\right)\boldsymbol{v}_{ij} \cdot \nabla_i W_{ij} + \frac{1}{\rho_i}\eta \boldsymbol{J}_i^2 \quad (18)$$

$$P_i = (\gamma - 1)\rho_i u_i \quad (19)$$

In addition, a divergence B correction term to maintain the divergence constraint of the plasma ($\nabla \cdot \boldsymbol{B} = 0$) [9], and several artificial dissipation terms that capture the shock and reduce numerical instability [10] are added. This physics model is incorporated into a multi-physics SPH code (named as SOPHIA) developed at Seoul National University [11].

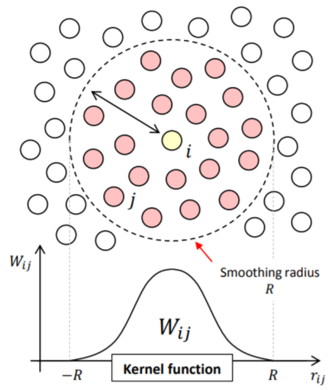

Fig. 1. Particle distribution with kernel function

4. Test Simulation

In this study, the simulations using the implemented models are conducted for three benchmark cases; (1) Brio & Wu shock tube (ideal-MHD), (2) resistive MHD shock simulation, and (3) magnetized Noh Z-pinch problem. The simulation results are compared with some reference Eulerian MHD simulations and analytical solutions.

4.1. Brio & Wu Shock Tube

The Brio & Wu (1988) shock tube problem generalizes the classic hydraulics Sod shock tube to magneto-hydrodynamics [12]. The problem consists of two initial states (to the left and right of the origin) brought into contact at t=0. The left state is initialized as $(\rho, v_x, v_y, v_z, B_y, B_z, P)$ = (1,0,0,1,0,1) and the right state (0.125,0,0,-1,0,0.1). This example tests whether the code can accurately represent the shocks, rarefactions, contact discontinuities, and the compound structures of MHD. For this reason, this problem has widely used as a benchmark problem to validate ideal MHD calculations [10, 13, 14]. In this study, 2D Brio&Wu shock tube problem is tested with 1000×50 particles in the range $x \in [-0.5,0.5]$.

We obtained 6 physical quantities (density, pressure, x,y-directional velocity, internal energy, and magnetic field) of the Brio&Wu shock problem through the SOPHIA code. Fig. 2 shows the simulation results in the Brio&Wu shock tube at 0.1 sec. The physical quantities of all particles onto the x-axis are shown as black dots, while the red lines show the numerical solution obtained from a proven Riemann solver [13]. As a result, the SOPHIA code performs numerically accurate simulations about the ideal MHD problem. We also have confirmed the effect of some additional terms such as the correction terms to satisfy $\nabla \cdot \boldsymbol{B}$ constraints, and artificial dissipation terms to handle shocks through this problem.

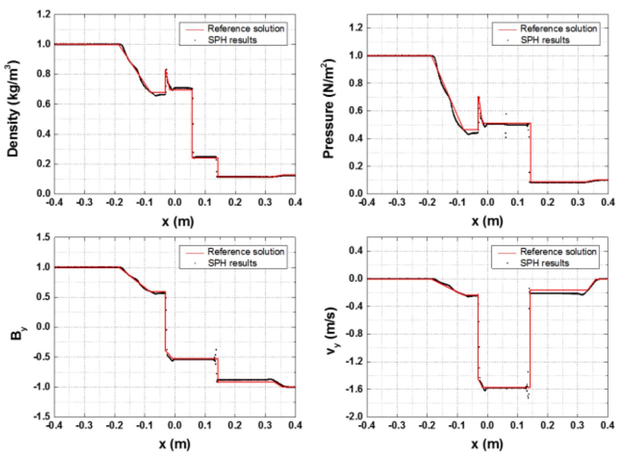

Fig. 2. Brio&Wu shock tube simulation results

4.2. Resistive MHD Shock Tube

In the actual pinch plasma simulations, the effect of plasma resistivity on plasma behavior must also be considered. Therefore, the resistive term is added in the induction equation and the energy equation, and the current density is additionally calculated in the SOPHIA code. In order to verify that the added terms work correctly, we perform the resistive MHD shock tube problem in which plasma resistivity is distributed in the Brio&Wu shock tube. Fig. 3 and 4 show the distribution of some physical quantities obtained when plasma resistivity is constant ($\eta = 1$) and it is a function of density($\eta = 10^{-3}\rho^{-2}$), respectively. In the case where resistivity is constant, the initial condition is $(\rho^L, v_x^L, P^L, \rho^R, v_x^R, P^R)$ = (1,0.4,1, 0.2,0.4,0.1), and in the case where the resistivity varies with density, the initial condition is $(\rho^L, v_x^L, P^L, \rho^R, v_x^R, P^R)$ = (1,1,0.5, 0.125,0.1,-0.5). As a result, the simulation shows very good agreement with the reference Eulerian code data. Interestingly, in the resistive MHD simulations, it is confirmed that stable calculations are performed without using the artificial resistivity term which contributed greatly to numerical stabilization in the ideal MHD simulations. We interpret this is because the actual plasma resistivity has a greater effect on its behavior than the artificial resistivity term.

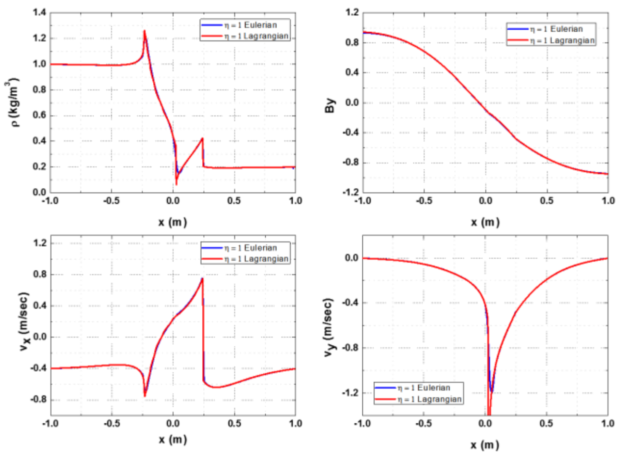

Fig. 3. Resistive MHD simulation results (η=1)

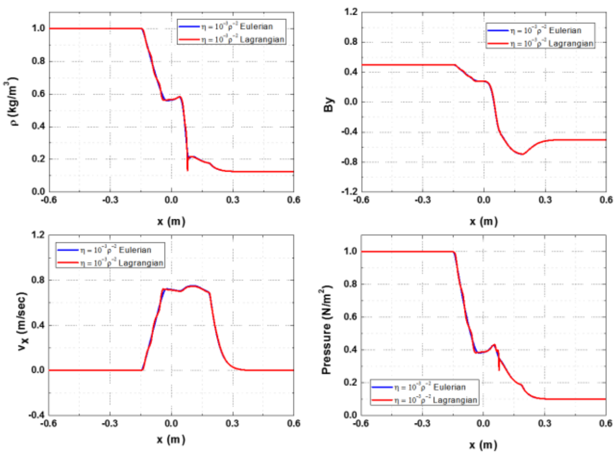

Fig. 4. Resistive MHD simulation results (η=$10^{-3}\rho^{-2}$)

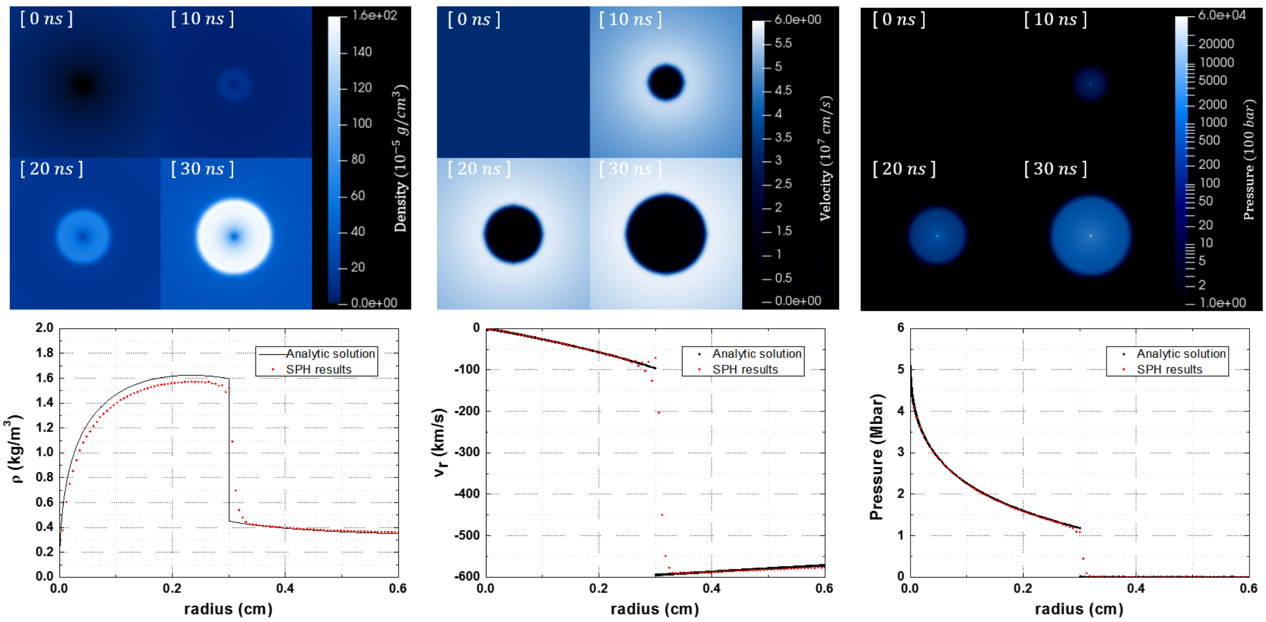

Fig. 5. Magnetized Noh Z-pinch simulation results from 2-dimensional SPH code

4.3. Magnetized Noh Z-pinch Problem

A. L. Velikovich (2011) proposed the Magnetized Noh problem as an example to verify the ability to analyze pinch plasma [15]. Noh problem has been used over the years for verification of codes designed to deal with implosion such as in inertial confinement fusion, to test the hydrodynamic component of MHD codes. The extension of this classic gas dynamics Noh problem to the electromagnetic problem is the Magnetized Noh Z-pinch problem. The operation of a Z-pinch is very simple. A current driven through a cylindrical column of the plasma induces the material to rapidly compress axially through J × B force. In this problem, the initial properties of plasma are expressed as a function of r, the distance from the central axis. ($\rho = 3.1831 \times 10^{-5} r^2$ [g/cm^3], $v_r = -3.24101 \times 10^7$ [cm/*sec*] , $B_\phi = 6.35584 \times 10^8 r$ [gauss] , $P = C \times B_\phi^2$) In this case, β, which represents the ratio of the plasma pressure and the magnetic pressure, is $8\pi \times 10^{-6}$. Fig. 5 is the comparison of the analytic solution and the results obtained through SOPHIA code for magnetized Noh problem. As a result, it is found that the implemented model predicts the pinch plasma behaviors fairly well.

5. Summary

In this study, a resistive MHD based SPH code is developed for the simulation of electrically conducting fluids, especially pinch plasma. This non-ideal MHD code is configured by sequentially adding terms necessary for pinch simulation to the existing SPH-based hydrodynamics code. For the verification of the model, three benchmark simulations are performed using the implemented code; (1) Brio & Wu shock tube (ideal-MHD), (2) resistive MHD shock simulation, and (3) magnetized Noh Z-pinch problem. The simulations are compared with some reference Eulerian MHD simulations and analytical solutions. The results of the simulations quantitatively and qualitatively show that SOPHIA code well simulates the behavior of the plasma including a simple Z-pinch problem.

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