Mathematical Justification for Large Mass Method for Dynamic Responses of Two D.O.F. System Excited by Ground Acceleration Time History

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1. Introduction

According to reference [1], there are two numerical methods for dynamic analysis of a structure subjected to ground acceleration. One is the large mass FEM simulation technique and the other is the large stiffness FEM simulation technique. The numerical techniques are easily incorporated into a commercial program to solve the problem. However, many users still debate on the validity of the large mass method because its analytical background has not been provided sufficiently, yet.[2] The questions often asked by users are as follows. Is it a standard method of using an acceleration time history by applying a force to the large mass? How do you calculate mass participation ratio or how sensitive is the magnitude of the large mass that is used? When does a problem with round-off error start to kick in?

In this paper, the mathematical aspects of a two degrees of freedom discrete model are investigated to answer the questions.

2. Mathematical Analysis

2.1 Original Model and AF Model

Consider the two-degree-of-freedom system shown in Fig. 1.

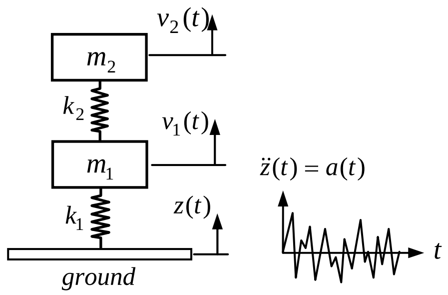

Fig. 1. Original model with all the initial conditions zero.

The system equation of motion is given as follows:

$$\begin{cases} \ddot{z} = a \\ m_1\ddot{v}_1 + k_1(v_1 - z) + k_2(v_1 - v_2) = 0 \\ m_2\ddot{v}_2 + k_2(v_2 - v_1) = 0 \end{cases} \qquad (1)$$

Eq. (1) can be rewritten as

$$\begin{cases} m_1\ddot{w}_1 + (k_1 + k_2)w_1 - k_2 w_2 = -m_1 a \\ m_2\ddot{w}_2 - k_2 w_1 + k_2 w_2 = -m_2 a \end{cases} \qquad (2)$$

where $w_1 = v_1 - z$ and $w_2 = v_2 - z$.

Eq. (2) can be interpreted as the equation of the system in Fig. 2. This model is called 'AF model' in this paper.

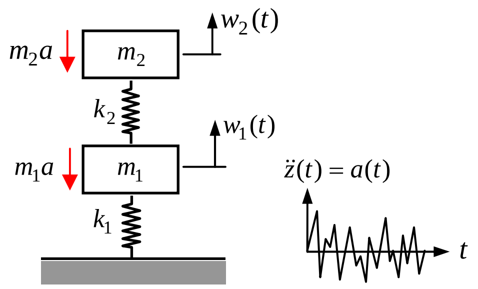

Fig. 2. AF (Additional Force) model.

2.2 Large Mass Model

For the ground to move in accordance with the given acceleration $a(t)$, many users employ a large mass for the ground and apply an external force to it as shown in Fig. 3.

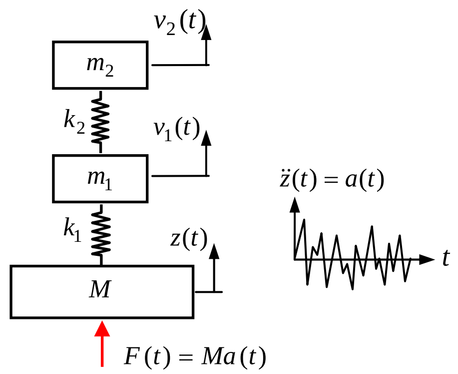

Fig. 3. LM (Large Mass) model.

The equation of motion of the LM model in Fig. 3 is given by

$$\begin{cases} M\ddot{z} + k_1 z - k_1 v_1 = Ma \\ m_1\ddot{v}_1 - k_1 z + (k_1 + k_2)v_1 - k_2 v_2 = 0 \\ m_2\ddot{v}_2 - k_2 v_1 + k_2 v_2 = 0 \end{cases} \qquad (3)$$

Summing the three equations, we have

$$M\ddot{z} + m_1\ddot{v}_1 + m_2\ddot{v}_2 = Ma \qquad (4)$$

or

$$\frac{\ddot{z}}{a} + \frac{\ddot{v}_1}{(M/m_1)a} + \frac{\ddot{v}_2}{(M/m_2)a} = 1, \qquad (5)$$

where $a \neq 0$. Eq. (5) is plotted in $\ddot{v}_1 - \ddot{v}_2 - \ddot{z}$ space, which is shown in Fig. 4.

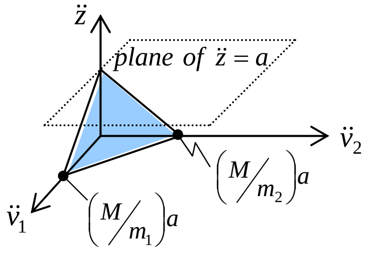

Fig. 4. The plane of equation (5).

Let us define a parameter called mass ratio:

$$\alpha = M/\max(m_1,\ m_2), \tag{6}$$

where $\max(m_1, m_2)$ denotes the maximum among the structural masses in the parentheses.

Assume that $a(\neq 0)$ is a finite value. Then $\ddot{z}$ and $\ddot{v}_1$, $\ddot{v}_2$ have finite values. Then, following theorems are drawn easily from the Eq. (5) and Fig. 4:

[Theorem 1]

If $\alpha \to \infty$, then $\ddot{z} \to a$.

Theorem-1 says that α should be large for $\ddot{z}$ to approach the given ground acceleration a . However, it does not say how much large the mass is.

[Theorem 2]

If $\alpha \to \infty$ and $\ddot{z} \to a$, then

$$\frac{\ddot{v}_1}{(M/m_1)a} + \frac{\ddot{v}_2}{(M/m_2)a} \to 0 . \tag{7}$$

Theorem-2 implies that, for an appropriately large value of α , Eq. (7) can be satisfied approximately without $\ddot{v}_1 = \ddot{v}_2 = 0$ if $\ddot{v}_1 << (M/m_1)a$ and $\ddot{v}_2 << (M/m_2)a$.

[Theorem 3]

If $\alpha \to \infty$ and $\ddot{v}_1 = \ddot{v}_2 = 0$ in Eq. (5), then $\ddot{z} = a$.

If $\ddot{v}_1 = \ddot{v}_2 = 0$, $\ddot{z}$ will be the same as the ground acceleration from Eq. (5). Such a situation can occur due to numerical errors in the computer. Theorem-3 implies that too large value of α could produce zero acceleration of structure with aid of numerical error. The numerical error can occur due to the finite number of bits. So the upper limit of α depends on machines or programs.

[Theorem 4]

If the mass ratio of α is a small finite value, then $\ddot{z} \neq a$ unless $\ddot{v}_1 = \ddot{v}_2 = 0$.

The above theorems say that the accuracy of the solutions of LM model depends on the magnitude of the mass ratio. Too large or too small mass ratio will cause significant error. An appropriately large mass ratio, which is between small one and very large one, yields good approximate solutions, which can be is proved as follows:

The second and the third equations in Eq. (3) are:

$$\begin{cases} m_1\ddot{v}_1 + k_1(v_1 - z) + k_2(v_1 - v_2) = 0 \\ m_2\ddot{v}_2 + k_2(v_2 - v_1) = 0 \end{cases} \tag{8}$$

Subtracting $m_1\ddot{z}$ and $m_2\ddot{z}$ from the both sides of the above equations, respectively,

$$\begin{cases} m_1(\ddot{v}_1 - \ddot{z}) + k_1(v_1 - z) + k_2(v_1 - v_2) = -m_1\ddot{z} \\ m_2(\ddot{v}_2 - \ddot{z}) + k_2(v_2 - v_1) = -m_2\ddot{z} \end{cases} \tag{9}$$

or

$$\begin{cases} m_1\ddot{w}_1 + k_1w_1 + k_2(w_1 - w_2) = -m_1\ddot{z} \\ m_2\ddot{w}_2 + k_2(w_2 - w_1) = -m_2\ddot{z} \end{cases} \tag{10}$$

When $\ddot{z} \to a$, Eq. (10) can be rewritten as

$$\begin{cases} m_1\ddot{w}_1 + k_1w_1 + k_2(w_1 - w_2) \cong -m_1a \\ m_2\ddot{w}_2 + k_2(w_2 - w_1) \cong= -m_2a \end{cases} \tag{11}$$

Eq. (11) is the approximate form of Eq. (2) of AF model, that is, the solution of LM model is the approximate solution of AF model if we employ an appropriately large mass ratio. This indicates that an appropriately large mass ratio yields good approximate solutions.

[Theorem 5]

For an appropriately large mass ratio, the LM model produces good approximate solutions.

3. Conclusions

This paper shows that the large mass method can be employed as a standard method of using an acceleration time history by applying a force to the large mass. Strictly speaking, the large mass method produces approximate solutions if an appropriately large mass ratio is input.

REFERENCES

[1] J. T. Chen, et al., “Integral Presentation and Regularization for a Divergent Series Solution of a Beam Subjected to Support Motions,” Earthquake Engineering and Structural Dynamics, Vol. 23, pp. 909-925, 1996.

[2] Y.-W. Kim and M. J. Jhung, “Mathematical Analysis on Two Modeling Techniques for Dynamic Response of a Structure Subjected to a Ground Acceleration,” ASME 2010 PVP Conference, Paper No. PVP2010-25394, 2010.